Reachability Problems in Low-dimensional Nondeterministic Polynomial Maps over Integers

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Abstract

We study reachability problems for various nondeterministic polynomial maps in $\mathbb{Z}^n$. We prove that the reachability problem for very simple three-dimensional affine maps (with independent variables) is undecidable and is PSPACE-hard for both two-dimensional affine maps and one-dimensional quadratic maps. Then we show that the complexity of the reachability problem for maps without functions of the form $\pm x + a_0$ is lower. In this case the reachability problem is PSPACE for any dimension and if the dimension is not fixed, then the problem is PSPACE-complete. Finally we extend the model by considering maps as language acceptors and prove that the universality problem is undecidable for two-dimensional affine maps.

1. Introduction

Many iterative maps can exhibit complex and unpredictable dynamics. They appear in various parts of mathematics and in particular they have been extensively studied in the context of chaos theory [11, 17] and control theory [6, 5] where these maps have been defined over rational, real or complex numbers. On the other hand iterative maps over integers are also important in computer science as they can be seen as the simplest form of computer programs describing updates on integer counters or variables. The classical reachability problems (i.e., whether a given set of values in the counters/variables can be reached via iterations in loops) are in the core of verification procedures and the complexity of their solutions could vary depending on several factors such as the *type of iterative functions* (affine, linear, polynomial, elementary, etc.), the *form of maps* (i.e., deterministic, nondeterministic), the *number of variables* (i.e., dimension of a system) and even *history dependence* (i.e., when the next value depends on several previous values of counters/variables) [36, 37].

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[1] The author was supported by Royal Society Leverhulme Trust Senior Research Fellowship SRF\R1\201074 and EPSRC grant EP/M00077X/1

In this paper we study the decidability of the reachability problem for simple stateless systems of nondeterministic iterative polynomial maps defined on integer valued vectors. The simplest form of a map with polynomial updates, i.e., where the updates are of the form $\vec{x} = \vec{x} + \vec{a_0}$, can be seen as a vector addition system (VAS) on $\mathbb{Z}^n$. In this case the reachability problem in one-dimensional system (i.e., with one variable/counter) for additive updates of the form $x + a_0$ can be reduced to solving a single linear Diophantine equation with solutions over natural numbers and the generalisation to the multidimensional case, which is in the form of the n-dimensional vector addition system on $\mathbb{Z}^n$, is known to be NP-complete [10, 19].

A simple generalisation to two-dimensional system of affine updates instead of additive maps makes the reachability problem undecidable over rational numbers $\mathbb{Q}^2$; see [2]. The affine transformations, that can encode the Post correspondence problem, are of a very restricted form $p(x, y) = (q_1 x + q_2 y + q_3, q_4 y)$.

Taking it into account we are focusing on a natural generalisation by defining multidimensional system with n-variables over $\mathbb{Z}^n$, where each coordinate dependent only on one variable and then we study the complexity of the reachability problem for different types of the polynomial updates of the form

$$p(x_1, \ldots, x_n) = (p_1(x_1), \ldots, p_n(x_n))$$

for some univariate polynomials $p_i(x)$.

Our research is revealing many surprising complexity results. For example even a simple increase from additive to affine updates where each variable is only self-dependent leads to undecidability of the reachability problem in $\mathbb{Z}^3$. The core element of the proof is in the construction of affine functions that simulate a state structure and can be used to control in which order affine updates have to be applied. Then by generalising the construction to encode any graph structure in one of the dimension we show that in two-dimensional systems with affine functions the reachability is at least PSPACE-hard while we do not know whether in this case the problem is decidable or not. Using a different encoding technique we prove that in one-dimensional maps, the reachability problem is PSPACE-complete. In the encoding a configuration of a machine (in this case LBA) is stored as a residue class and updated accordingly using quartic polynomials. The same encoding idea was used in [16]. See Table 1 for the summary of the results.

degree / dim.	1		2	3	4
	the leading coefficient				
	$a_1 = \pm 1$	$a_1 \in \mathbb{Z}$			
1	NP-c. [19]	NP-h. [19]/PSPACE [16]			PSPACE-c.
2		PSPACE-h./?			
3		undecid.			

Table 1: Complexity of reachability problems in nondeterministic polynomial maps according to the degrees. Our results are on grey background.

Then we investigate the restrictions on the functions further by considering maps without functions of the form $\pm x + a_0$. Surprisingly, already for affine maps, this leads to NP-hardness for any fixed dimension as well as PSPACE-hardness if the dimension is not fixed. As for the upper bound, even for maps with polynomial updates — as long as updates of the form $\pm x + a_0$ are not present — the reachability problem is in PSPACE, and thus PSPACE-complete if the dimension is not fixed. This heavily contrasts our knowledge on general maps, where most notably, dimension two has no known upper bounds; see Table 2.

<table>
<tr><th rowspan="2">type / dim.</th><th colspan="2">affine</th><th colspan="2">polynomial</th></tr>
<tr><th>$a_1 \neq \pm 1$</th><th>$a_1 \in \mathbb{Z}$</th><th>not including $\pm x + a_0$</th><th>including $\pm x + a_0$</th></tr>
<tr><td>1</td><td rowspan="4">NP-h./PSPACE</td><td>NP-h. [19]/PSPACE [16]</td><td rowspan="4">NP-h./PSPACE</td><td>PSPACE-c.</td></tr>
<tr><td>2</td><td>PSPACE-h./?</td><td>PSPACE-h./?</td></tr>
<tr><td>3</td><td rowspan="3">undecid.</td><td rowspan="3">undecid.</td></tr>
<tr><td>$\vdots$</td></tr>
<tr><td>n</td><td>PSPACE-c.</td><td>PSPACE-c.</td></tr>
</table>

Table 2: Complexity of reachability problems in affine and polynomial maps with respect to inclusion of polynomials of the form $\pm x + a_0$. Our results are on grey background.

Finally, we take a more language-theoretic approach and consider maps as language acceptors. To this end, we fix the initial and target values, $\vec{z}_0$ and $\vec{z}_f$, in the reachability problem for polynomial maps. Then, we attach a letter over a finite alphabet Σ to each function. Now, a word $w \in \Sigma^*$ is accepted if there is a computation path from $\vec{z}_0$ to $\vec{z}_f$ reading w in the map. From this point of view, the reachability problem is the language emptiness problem. We study another natural language-theoretic question known as the *universality problem*, where we are asked whether all finite words are accepted by the map. We show that for two-dimensional affine maps the universality problem is undecidable by simulating an integer weighted automaton [20].

The research work on such systems with polynomial updates attracted visible attention in verification community indicating the lack of understanding of even such simple systems. In contrast to stateless systems (iterative maps) a model of *polynomial register machines* which have additional state structure has been studied in [16, 35]. The authors showed that the reachability problem is PSPACE-complete for one-dimensional polynomials and undecidable for two-dimensional polynomials, respectively.

Multidimensional linear polynomial iteration has also been considered from a different aspect. The vector reachability problem for n-dimensional matrices over $\mathbb{F}$, where $\mathbb{F} = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \ldots$, studies whether for given two vectors $\vec{x}_0, \vec{x}_f \in \mathbb{F}^n$ and a set of matrices $\{M_1, \ldots, M_k\} \subseteq \mathbb{F}^{n \times n}$, there exists a finite sequence of matrices such that $M_{i_1} \cdots M_{i_j} \vec{x}_0 = \vec{x}_f$. Since transforming a vector by a matrix can be expressed as a system of linear equations, the multidimensional linear polynomial iteration can be seen as a vector reachability problem. The main difference from our consideration is that we consider only polynomials of the form $p(x_1, \ldots, x_n) = (p_1(x_1), \ldots, p_n(x_n))$ for some univariate polynomials $p_i(x)$, while the polynomials in the vector reachability problem are of the form $p(x_1, \ldots, x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n, \ldots, a_{n1}x_1 + \cdots + a_{nn}x_n)$. The vector reachability problem has been proven to be undecidable for 6 three-dimensional integer matrices in [22] and for two 11-dimensional integer matrices in [21].

Recently, in [7], the authors studied a different variant of polynomial register machines. They restricted the considerations to affine updates and moreover, that the matrix monoid generated by the transformations is finite. In this variant, the reachability problem is reducible to the reachability problem for integer VASS [10, 19] and thus decidable. Additionally, the authors considered different restrictions on the affine updates and showed that some variants are PSPACE-complete while others are NP-complete.

The next level of complexity of iterative systems is to consider iterative maps over rationals, where even for one-dimensional affine maps the reachability problem is still open and can be seen as a special case of vector reachability for 2×2 matrix semigroups over $\mathbb{Q}$. However, for nondeterministic iterative polynomials maps of degree at least two, the reachability problem is decidable by an analysis of ρ-adic weights instead of Euclidean distances [9]. The solution of reachability for iterative one-dimensional maps over $\mathbb{Q}$ can particularly answer another long standing open problem — the reachability for piecewise affine maps (PAM). Iteration of deterministic one-dimensional piecewise affine map $f : \mathbb{Q} \to \mathbb{Q}$ is often used as a reference model to show hardness of reachability problems for various types of hybrid systems, where

$$f(x) = a_i \cdot x + b_i, \text{ for } x \in I_i, \ i = 1, \dots, k.$$

Here, $I_i = (l_i, r_i]$ is a finite set of disjoint intervals and all coefficients a_i, b_i and the extremities of I_i are from $\mathbb{Q}$. The reachability problem for PAMs is "Given a PAM $f(x)$ and two points x_0, $y \in \mathbb{Q}$. Decide whether exist $n \in \mathbb{N}$ such that $f^n(x_0) = y$?" For the special case of "complete maps" where each subinterval of the unit circle I_i is mapped to the unit circle the reachability can be reduced to the reachability of nondeterministic affine maps, i.e, tracing back non-deterministic trajectories with inverse functions (also affine) starting from y [9]. Piecewise maps and related reachability problems have been studied extensively in [28, 3, 29, 9] and can be generalised to non-deterministic maps if the intervals are not disjoint or, in more general settings, when more than one function is assigned to the same interval, that is, should be applied nondeterministically. Thus our iterative maps can be seen as trivial piecewise nondeterministic maps where all functions have been assigned only to one interval $(-\infty, \infty)$.

Already starting from dimension two, the reachability problem is undecidable for deterministic two-dimensional piecewise affine maps over $\mathbb{Q}$. The reachability problem for one-dimensional piecewise affine maps is an open problem even when there are only two intervals. On the other hand, for more general updates the problem for deterministic maps is undecidable. For example, if the updates are based on the elementary functions $\{x^2, x^3, \sqrt{x}, \sqrt[3]{x}, 2x, x+1, x-1\}$ or on rational functions of the form $p(x) = \frac{ax^2+bx+c}{dx+e}$, where the coefficients are rational numbers [30], then the problem is undecidable.

The present paper analyses polynomial maps over integers focusing on the core power of iterative maps in minimalistic settings of integer numbers. A natural extension in the future is to consider maps over rationals or reals. All our lower bounds can be lifted to the more general settings. Moving to rationals or reals introduces on one hand new challenges. On the other hand it can quickly enrich the model leading to simpler undecidability constructions due to extra information that can be stores and used in case of rational numbers or leading to unpredictability due to the nature of reals. In particular, in Theorem 17, we show the undecidability of the reachability problem for specific three-dimensional rational maps over rationals while the corresponding maps over integers have a decidable reachability problem. When considering maps over rationals, the questions can become harder having connections to group theory [14] and number theory [8]. Changing the question from point reachability to staying within a set [15] or set reachability [1], can be made more amenable to analysis.

Also our approach to look at the reachability from language-theoretic point of view is in line with a recent paper [4] on the complexity of the zeroness problem for deterministic polynomial automata over rationals (i.e., a problem whether a given automaton outputs zero on all words). Authors of [4] showed that while this problem is non-primitive recursive in general, there is a subclass of polynomial automata for which the zeroness problem is primitive recursive.

This paper is the full version of conference papers [33] and [27]. Unlike the conference versions, we give full proofs of the results and provide additional comments on the constructions.

This paper is organised as follows. In the next section, we introduce basic definitions and models used in the paper. In Section 3, we prove that the reachability problem for one-dimensional quartic maps is PSPACE-hard. Then, in Section 4, we consider the reachability problem for multidimensional maps. Namely we show that the problem is undecidable for three-dimensional affine maps and PSPACE-hard for two-dimensional affine maps. In Section 5, we study maps without functions of the form $\pm x + a_0$ and show that the reachability problem is PSPACE for any dimension and that if the dimension is not fixed, then the problem is PSPACE-complete. Finally in Section 6, we extend the model by considering maps as language acceptors and prove that the universality problem is undecidable for two-dimensional affine maps.

2. Preliminaries

We denote the set of natural numbers, integers and rational numbers by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$, respectively. The integers are assumed to be encoded in binary. For $z_1 \leq z_2 \in \mathbb{Z}$, we denote the *closed interval* by $[z_1, z_2] = \{z \in \mathbb{Z} \mid z_1 \leq z \leq z_2\}$.

We denote the ring of polynomials with integer coefficients over a variable x by $\mathbb{Z}[x]$. A polynomial $p(x) \in \mathbb{Z}[x]$ is $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_i \in \mathbb{Z}$ and $n \geq 0$. We represent polynomials in sparse encoding by a sequence of pairs $(i, a_i)_{i \in I}$, where $I = \{i \in \{0, \ldots, n\} \mid a_i \neq 0\}$. Deciding whether for a given $y \in \mathbb{Z}$, the polynomial $p(y)$ evaluates to a positive number can be done in polynomial time [12]. The multidimensional polynomials $\mathbb{Z}[\vec{x}]^n$ with independent variables $\vec{x} = (x_1, \ldots, x_n)$ are of the form $p(x_1, \ldots, x_n) = (p_1(x_1), \ldots, p_n(x_n))$ for some univariate polynomials $p_1(x), \ldots, p_n(x) \in \mathbb{Z}[x]$. That is, dimensions are independent and do not affect the values in other dimensions.

Throughout the paper, we investigate the reachability problem for different classes of polynomials and in order to simplify the terminology, we give names to the commonly used classes:

$$\begin{aligned}
\text{Additive polynomials:} &\quad \mathsf{Add}_{\mathbb{Z}} = \{\pm x + a_0 \mid a_0 \in \mathbb{Z}\},\\
\text{Affine polynomials:} &\quad \mathsf{Aff}_{\mathbb{Z}}[x] = \{a_1 x + a_0 \mid a_1, a_0 \in \mathbb{Z}\},\\
\text{Quadratic polynomials:} &\quad \mathsf{Quad}_{\mathbb{Z}}[x] = \{a_2 x^2 + a_1 x + a_0 \mid a_2, a_1, a_0 \in \mathbb{Z}\},\\
\text{Quartic polynomials:} &\quad \mathsf{Quart}_{\mathbb{Z}}[x] = \{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \mid a_4, a_3, a_2, a_1, a_0 \in \mathbb{Z}\}.
\end{aligned}$$

We also define the classes of polynomials without additive polynomials[2], i.e., updates

[2] In [16], additive polynomials were called counter-like as they are similar to updates in counter machines and VASSs. They could also be called (generalised) monic linear polynomials but the authors find the name cumbersome.

of form $\pm x + a_0$:

$$\begin{aligned}\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}} &= \{a_1 x + a_0 \in \mathsf{Aff}_{\mathbb{Z}}[x] \mid a_1 \neq \pm 1\},\\ \mathbb{Z}[x] \setminus \mathsf{Add}_{\mathbb{Z}} &= \{p(x) \in \mathbb{Z}[x] \mid p(x) \neq \pm x + a_0, \text{ where } a_0 \in \mathbb{Z}\}.\end{aligned}$$

The multidimensional variants $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n$, $\mathsf{Quad}_{\mathbb{Z}}[\vec{x}]^n$ and $\mathsf{Quart}_{\mathbb{Z}}[\vec{x}]^n$ are defined in the natural way, while $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is defined as $((\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}) \cup \{x\}) \times \cdots \times ((\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}) \cup \{x\}))$ and $\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is defined analogously. The class $\mathbb{Z}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ seems artificial at first but, as we prove later, polynomials of the form $\pm x + a_0$ play a vital role in whether the reachability problem is decidable or undecidable. Indeed, we will show that for $\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$, the reachability problem is in PSPACE, while the problem is undecidable already for $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^3$. Note that the multidimensional variant allows a particular additive polynomial in each component. Namely, the identity polynomial $p(x) = x$. This polynomial is essential as it allows the map to either apply a polynomial of $\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ or $\mathbb{Z}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ or keep the value the same in each component. This is a crucial property for the lower bounds proven in Lemma 18 and subsequent results relying on the lemma. The different classes of polynomials are depicted below:

$$a_n x^n + \cdots + \underbrace{a_4 x^4 + a_3 x^3 + \overbrace{a_2 x^2 + \underbrace{a_1 x + a_0}_{\in\, \mathsf{Aff}_{\mathbb{Z}}[x]}}^{\in\, \mathsf{Quad}_{\mathbb{Z}}[x]}}_{\in\, \mathsf{Quart}_{\mathbb{Z}}[x]} \in \mathbb{Z}[x] \qquad\qquad \pm x + a_0 \in \mathsf{Add}_{\mathbb{Z}}$$

In the encoding of Section 3, we use the Chinese remainder theorem to find the unique solution to a system of linear congruences. That is, for given pairwise co-prime positive integers $n_1, \ldots, n_k$ and $b_1, \ldots, b_k \in \mathbb{Z}$, the system of linear congruences $x \equiv b_i \mod n_i$ for $i = 1, \ldots, k$ has the unique solution modulo $n_1 \cdots n_k$. Recall that a residue class b modulo n is the set of integers $\{\ldots, b-n, b, b+n, \ldots\}$.

An *n-dimensional polynomial register machine* (n-PRM) is a tuple $\mathcal{R} = (Q, \Delta)$, where Q is a finite set of states and $\Delta \subseteq Q \times \mathbb{Z}[\vec{x}]^n \times Q$ is the set of transitions labelled by polynomials with variable $\vec{x} \in \mathbb{Z}^n$. A *configuration* of $\mathcal{R}$ is a tuple $[q, \vec{x}] \in Q \times \mathbb{Z}^n$. A configuration $[q', \vec{y}']$ is *reachable* from a configuration $[q, \vec{y}]$ by a transition $(q, f(\vec{x}), q')$ if $f(\vec{y}) = \vec{y}'$. This is denoted by $[q, \vec{y}] \to_{\mathcal{R}} [q', \vec{y}']$. The reflexive and transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to_{\mathcal{R}}^*$. The *reachability problem* is, given two configurations c and c' of $\mathcal{R}$, to decide whether $c \to_{\mathcal{R}}^* c'$ holds. If an n-PRM $\mathcal{R}$ has only one state, then $\mathcal{R}$ is called a *nondeterministic polynomial map* or a map over $\mathbb{Z}[\vec{x}]^n$ for short.

An n-PRM is called an *n-dimensional affine register machine* (n-ARM) if every transition is labelled by an affine polynomial. A *nondeterministic affine map* (a map over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n$) is defined analogously.

A *linear-bounded automaton* (LBA) is a Turing machine with a tape bounded by a linear function of the length of the input. In other words, an LBA can be viewed as a Turing machine with a finite tape. We denote an LBA $\mathcal{A}$ by a tuple (Q, Γ, δ), where Q is a finite set of states, $\Gamma = \{\rhd, \lhd, 0, 1\}$ is a finite tape alphabet, which includes two special symbols $\rhd$ and $\lhd$ serving as left and right endmarkers of the tape. The *transition function* is a relation $\delta \subseteq Q \times \Gamma \times Q \times \Gamma \times \{L, R\}$ such that a transition $(q, \gamma, q', \gamma', d) \in \delta$ implies that if the LBA $\mathcal{A}$ is in state q and reads γ at the current head position on the tape then $\mathcal{A}$ moves to the state q' while writing γ' onto the tape and moving the head position according to the direction d. Note that the head moves to the right if $d = R$ and to the left if $d = L$. Since the tape of $\mathcal{A}$ is delimited by endmarkers, we have conditions that if the head is at the left endmarker $\rhd$, then d is always R and $\gamma' = \rhd$, analogously if $\gamma = \lhd$, then $d = L$ and $\gamma' = \lhd$. A *configuration* of $\mathcal{A}$ is a tuple $(q, \rhd w \lhd, i)$, where $q \in Q$ is the currents state, $w \in (\Gamma \setminus \{\rhd, \lhd\})^*$ is the tape content and $i \in [0, |w|+1]$ is the current head position. We define the successor relation $\to_{\mathcal{A}}$ between two configurations in the standard way and $\to_{\mathcal{A}}^*$ is the reflexive and transitive closure of $\to_{\mathcal{A}}$. See for example [23] for more details on LBA.

The reachability problem of a given LBA $\mathcal{A}$ is to decide whether for given length of the tape s, states q and q', $(q, \rhd 0^s \lhd, 0) \to_{\mathcal{A}}^* (q', \rhd 0^s \lhd, 0)$ holds and is well-known to be PSPACE-complete. We often enumerate the states of $\mathcal{A}$ such that $q' = q_{|Q|}$ and further assume that $\mathcal{A}$ enters state $q_{|Q|}$ only in configuration $(q_{|Q|}, \rhd 0^s \lhd, 0)$.

Given an *alphabet* $\Sigma = \{a_1, a_2, \ldots, a_m\}$, a finite *word* u is an element of semigroup Σ^*. That is $u = u_1 u_2 \cdots u_n$, where $u_i \in \Sigma$. The *empty word* is denoted by ε. The length of a finite word u is denoted by $|u|$ and $|\varepsilon| = 0$. We denote $w \in \Sigma^*$ with $|w| = s$ by $w \in \Sigma^s$.

Lemma 1. *Let $\Sigma = \{a, b\}$ and $w \in \Sigma^*$. Let $\tau : \Sigma \to \mathbb{N}$ be defined as $\tau(a) = 1$ and $\tau(b) = 2$. The function $\sigma : \Sigma^* \to \mathbb{N}$ defined by*

$$\sigma(w_1 w_2 \cdots w_k) = \sum_{i=1}^{k} \tau(w_i) \cdot 3^{k-i}$$

and $\sigma(\varepsilon) = 0$ is an injective function.

The function σ of the previous lemma, outputs a number in ternary representation of the input word over the binary alphabet Σ. Since each natural number has a unique ternary representation, σ is clearly an injective function.

We use the Post's correspondence problem (PCP) [34] to show undecidability in nondeterministic affine maps. Let $\Sigma = \{a, b\}$ be a binary alphabet and

$$P = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\} \subseteq \Sigma^* \times \Sigma^*$$

be a set of pairs of words where $n \geq 2$. Then, the PCP is to determine if there exists a finite sequence of indices $\ell_1, \ell_2, \ldots, \ell_k$ with each $1 \leq \ell_i \leq n$ such that

$$u_{\ell_1} u_{\ell_2} \cdots u_{\ell_k} = v_{\ell_1} v_{\ell_2} \cdots v_{\ell_k}.$$

The PCP was shown to be undecidable for $n = 7$ in [31]. Recently the undecidability bound has been improved to $n = 5$ [32]. We denote the minimal number of pairs for which the PCP is undecidable by n_p and use it to describe other undecidability bounds presented in the paper.

3. Reachability in maps over $\mathsf{Quart}_{\mathbb{Z}}[x]$

In this section, we prove that the reachability problem in one-dimensional polynomial maps with quartic polynomials is PSPACE-hard and PSPACE in general. The proof of the lower bound is similar to the proof of PSPACE-hardness of the reachability problem for polynomial register machines as found in [16]. Both proofs reduce from the reachability problem for LBA. Let us fix an LBA $\mathcal{A}$ with a tape of s letters for the remainder of the section. The main difference of the proofs is that in [16], the states of PRM contain partial information from a configuration of $\mathcal{A}$: the state of $\mathcal{A}$, the position of the read/write head and the letter that the head is currently reading, while the tape content is encoded as an integer and modified by the transitions according to the instructions of $\mathcal{A}$. In our proof, the whole configuration of $\mathcal{A}$ is encoded as an integer and updated according to the instructions of $\mathcal{A}$.

First, let us recall some definitions from [16]. Let p_i denote the $(i+3)$-th prime number, that is, $p_1 = 7, p_2 = 11, \ldots$ and let P be the product of m such primes, $P = \prod_{i=1}^{m} p_i$, where $m = s + s \cdot |Q|$.

The main idea of the encoding is to consider the integer line $\mathbb{Z}$ modulo P and integers as the corresponding residue classes. We are interested in the residue classes that satisfy linear congruences modulo p_i for different p_i. The first s primes will correspond to each cell of the tape and the next $s \cdot |Q|$ primes will correspond to the head being in a particular state in a particular cell. Note, that for the sake of simplicity, we omit the behaviour of the head on the border letters. In fact, it is quite easy to deal with these transitions as, among other information, we also encode the position of the head into our integer. Then it is easy to hard-code the behaviour of $\mathcal{A}$ on the border letters into corresponding polynomials. Indeed, consider a configuration $[q, 1, \triangleright aw\triangleleft]$ such that the next configuration is $[q', 0, \triangleright bw\triangleleft]$, where $a, b \in \{0, 1\}$ and $w \in \{0, 1\}^{s-1}$. As the head respects the border letters, the next configuration is $[q'', 1, \triangleright bw\triangleleft]$. That is, all we need to simulate is a transition from state q to q'' and possible rewriting of a to b.

We are not interested in all residue classes modulo P and only a tiny fraction of the residue classes is used to store information. A residue class r is of interest to us, if for every $1 \leq i \leq m$, there is some $b_i \in \{0, 1, 2\}$ such that $r \equiv b_i \mod p_i$. We call such residue class *sane* and denote the set of all sane residue classes by S. A configuration $[q_j, i, \triangleright w\triangleleft]$, where $i = \{1, \ldots, s\}$ and $w \in \{0, 1\}^s$, corresponds to a residue class r satisfying the system of congruence equations

$$\begin{aligned} r &\equiv w_1 \mod p_1, \\ r &\equiv w_2 \mod p_2, \\ &\vdots \\ r &\equiv w_s \mod p_s, \end{aligned} \qquad \begin{aligned} r &\equiv 1 \mod p_\ell \text{ if } \ell = s + j + (i-1)|Q|, \\ r &\equiv 0 \mod p_\ell \text{ if } \ell > s \text{ and } \ell \neq s + j + (i-1)|Q|. \end{aligned} \tag{1}$$

Intuitively, the first s congruence equations encode the tape content and the next $s|Q|$ equations encode the state and position of the head. We illustrate how a configuration $[q_3, 2, \triangleright 1001 \cdots 1\triangleleft]$ of an LBA corresponds to the residue class r satisfying the system of linear congruences (1) in Figure 1.

Since the head of an LBA modifies the tape locally, to simulate a transition, for example, $\delta(q_j, a) = (q_k, a', L)$, it is enough to check that the residue class r satisfies congruence equations

$$r \equiv 1 \mod p_{s+j+(i-1)|Q|} \qquad \text{and} \qquad r \equiv a \mod p_i$$

for some $i \in \{1, \ldots, s\}$. Then, the transition is simulated by moving from r to a residue

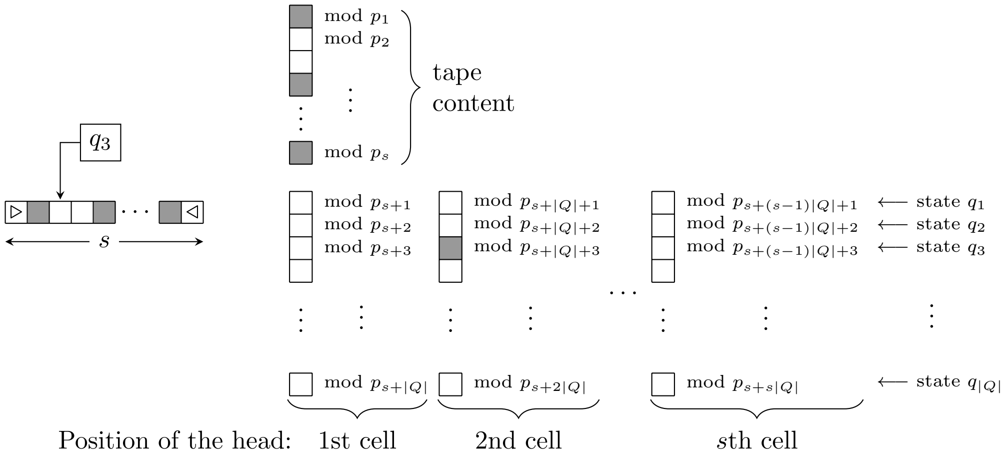

Figure 1: An illustration how configuration $[q_3, 2, \triangleright 1001 \cdots 1 \triangleleft]$ of an LBA (left) is encoded as residue class r satisfying a system of linear congruences. Here, letters 0 and 1 are represented by white and grey squares, respectively. A grey square in the ith cell column and the jth state row represents the head being in the ith cell in state q_j.

class r' satisfying congruence equations

$$\begin{aligned}
&r' \equiv 0 \mod p_{s+j+(i-1)|Q|},\\
&r' \equiv 1 \mod p_{s+k+(i-2)|Q|},\\
&r' \equiv a' \mod p_i,\\
&r' \equiv r \mod p_\ell \text{ for all}\\
&\qquad \ell \in \{1, \dots, s + s \cdot |Q|\} \setminus \{i, s+j+(i-1)|Q|, s+k+(i-2)|Q|\}.
\end{aligned}$$

That is, to simulate a transition of the LBA, first we need to check that the current residue class r corresponds to a configuration $[q_j, i, \triangleright w \triangleleft]$, where $w_i = a$, for some i and the other letters of w are irrelevant. Then we move to the residue class r' corresponding to the configuration $[q_k, i-1, \triangleright w' \triangleleft]$, where $w'_i = a'$ and $w'_\ell = w_\ell$, for all $\ell = \{1, \dots, s\} \setminus \{i\}$. Analogously, to simulate a transition where the head moves to the right, similar checks need to be performed.

To this end, we need to *locally* modify the residue classes. That is, we need to have a polynomial $p(x)$ such that $p(r) = r'$. There are three mappings that are defined for each index $i \in \{1, \dots, m\}$, $\textsc{flip}_i, \textsc{eqzero}_i, \textsc{eqone}_i : S \to S$. The mapping $\textsc{flip}_i$ is used to change the residue class of r modulo p_i in a particular way. The mappings $\textsc{eqzero}_i$ and $\textsc{eqone}_i$ are used to check that the residue class of r modulo p_i is zero or one, respectively.

Let us describe the mappings in more details. For the mapping $\textsc{flip}_i(r)$ there are three cases depending on whether $r \equiv 0, 1, 2 \mod p_i$:

$$\text{if } r \equiv 0 \mod p_i: \quad \textsc{flip}_i(r) \equiv \begin{cases} 1 & \mod p_i \\ r & \mod p_j \end{cases} \qquad \text{if } r \equiv 1 \mod p_i: \quad \textsc{flip}_i(r) \equiv \begin{cases} 0 & \mod p_i \\ r & \mod p_j \end{cases} \qquad \text{if } r \equiv 2 \mod p_i: \quad \textsc{flip}_i(r) \equiv \begin{cases} 2 & \mod p_i \\ r & \mod p_j, \end{cases}$$

where $j \neq i$.

Similarly, for the remaining two mappings, there are three cases depending on whether $r \equiv 0, 1, 2 \mod p_i$:

$$\text{if } r \equiv 0 \mod p_i : \qquad\qquad \text{if } r \equiv 1, 2 \mod p_i :$$

$$\textsc{eqzero}_i(r) \equiv \begin{cases} 0 & \mod p_i \\ r & \mod p_j \end{cases} \qquad \textsc{eqzero}_i(r) \equiv \begin{cases} 2 & \mod p_i \\ r & \mod p_j \end{cases}$$

$$\text{if } r \equiv 1 \mod p_i : \qquad\qquad \text{if } r \equiv 0, 2 \mod p_i :$$

$$\textsc{eqone}_i(r) \equiv \begin{cases} 1 & \mod p_i \\ r & \mod p_j \end{cases} \qquad \textsc{eqone}_i(r) \equiv \begin{cases} 2 & \mod p_i \\ r & \mod p_j, \end{cases}$$

where $j \neq i$.

The move $\delta(q_j, 0) = (q_k, 0, L)$ of LBA $\mathcal{M}$ when the head is in ith position is now realised by a composition of the functions

$$\textsc{flip}_{s+k+(i-2)|Q|} \circ \textsc{flip}_{s+j+(i-1)|Q|} \circ \textsc{eqzero}_i \circ \textsc{eqone}_{s+j+(i-1)|Q|}.$$

In Figure 2 we illustrate how moves $\delta(q_j, 0) = (q_k, 0, L)$ and $\delta(q_j, 1) = (q_k, 0, R)$ of LBA $\mathcal{A}$ are realised for the configuration $[q, i, \triangleright w \triangleleft]$. Note, that we do not assume that $q = q_j$ or that $w_i = 0$ for the first move or $w_i = 1$ for the second move. Mappings $\textsc{eqzero}_\ell$ and $\textsc{eqone}_\ell$ verify that the bit encoded in the residue class modulo p_ℓ is 0 or 1, respectively.

$\textsc{eqzero}_i$

$\delta(q_j, 0) = (q_k, 0, L)$

$\textsc{flip}_{s+k+(i-2)|Q|} \circ \textsc{flip}_{s+j+(i-1)|Q|} \circ \textsc{eqone}_{s+j+(i-1)|Q|}$

$\textsc{flip}_i \circ \textsc{eqone}_i$

$\delta(q_j, 1) = (q_k, 0, R)$

$\textsc{flip}_{s+k+i|Q|} \circ \textsc{flip}_{s+j+(i-1)|Q|} \circ \textsc{eqone}_{s+j+(i-1)|Q|}$

Figure 2: An illustration of mappings corresponding to moves of LBA.

The crucial ingredient for the simulation is that the functions $\textsc{flip}_i$, $\textsc{eqzero}_i$ and $\textsc{eqone}_i$ can be realised by polynomials with coefficients in $\{0, \ldots, P-1\}$. We present the lemma of [16].

Lemma 2. *For any $1 \leq i \leq m$ and any of $\textsc{flip}_i, \textsc{eqzero}_i, \textsc{eqone}_i : S \to S$, there is a quadratic polynomial with coefficients from $\{0, \ldots, P-1\}$ that realises the respective function.*

Proof. First, we show the polynomials corresponding to the mappings $\textsc{flip}_i$, $\textsc{eqzero}_i$ and $\textsc{eqone}_i$ that map the values correctly when considering only $\mathbb{Z}/p_i\mathbb{Z}$. Then we explain how to modify them to also map the values correctly for all $\mathbb{Z}/p_j\mathbb{Z}$, where $j \neq i$.

It is easy to verify that the polynomials

$$p_{eqzero}(x) = -x^2 + 3x,$$
$$p_{eqone}(x) = x^2 - 2x + 2 \text{ and}$$
$$p_{flip}(x) = 3 \cdot 2^{-1} x^2 - 5 \cdot 2^{-1} x + 1$$

realise the respective mappings. Note that since $p_i \geq 7$, 2 has a multiplicative inverse. For example, let $p_i = 11$, then $2^{-1} = \frac{p_i - 1}{2} = 6$ and the evaluations of polynomials $p_{eqzero}(x)$, $p_{eqone}(x)$ and $p_{flip}(x)$ are presented in Table 3.

x	$p_{eqzero}(x)$	$p_{eqone}(x)$	$p_{flip}(x)$
0	$-0^2 + 3 \cdot 0 \equiv \mathbf{0}$	$0^2 - 2 \cdot 0 + 2 \equiv \mathbf{2}$	$3 \cdot 6 \cdot 0^2 - 5 \cdot 6 \cdot 0 + 1 \equiv \mathbf{1}$
1	$-1^2 + 3 \cdot 1 \equiv \mathbf{2}$	$1^2 - 2 \cdot 1 + 2 \equiv \mathbf{1}$	$3 \cdot 6 \cdot 1^2 - 5 \cdot 6 \cdot 1 + 1 = -11 \equiv \mathbf{0}$
2	$-2^2 + 3 \cdot 2 \equiv \mathbf{2}$	$2^2 - 2 \cdot 2 + 2 \equiv \mathbf{2}$	$3 \cdot 6 \cdot 2^2 - 5 \cdot 6 \cdot 2 + 1 = 13 \equiv \mathbf{2}$

Table 3: Evaluations of polynomials $p_{eqzero}(x)$, $p_{eqone}(x)$ and $p_{flip}(x)$ in $\mathbb{Z}/11\mathbb{Z}$.

Although these polynomials realise the conditions of FLIP_i, EQZERO_i and EQONE_i for i, they (generally) do not realise the conditions when $j \neq i$. That is, $p_{eqzero}(x) \neq x$ when considering the polynomials in $\mathbb{Z}/p_j\mathbb{Z}$. To illustrate this, consider $p_{eqzero}(1)$ as above, but now with respect to $p_j = 7$. By the definition of EQZERO_i, it should remain unchanged, that is $p_{eqzero}(1) = 1$ with respect to p_j. This is not the case, as 1 and 2 are different residue classes. To obtain polynomials corresponding to FLIP_i, EQZERO_i and EQONE_i, we consider polynomials $p_{eqzero}(x)$, $p_{eqone}(x)$ and $p_{flip}(x)$ as $a_2x_2 + a_1x + a_0$ and construct a system of congruences for each $\ell = \{0, 1, 2\}$:

$$\begin{aligned} x &\equiv a_\ell \mod p_i \\ x &\equiv b_\ell \mod p_j \text{ for each } j \in \{1, \ldots, m\} \setminus \{i\}, \end{aligned}$$

where $b_1 = 1$ and $b_0 = b_2 = 0$. By applying the Chinese remainder theorem, we obtain the unique solution for each coefficient and obtain the polynomials $p_{eqzero,i}(x)$, $p_{eqone,i}(x)$ and $p_{flip,i}(x)$ by replacing the original coefficients with these unique solutions. ☐

Now, for each $i \in \{1, \ldots, m\}$ and each transition $\delta(q_j, a) = (q_k, a', D)$, where $a, a' \in \{0, 1\}$ and $D = \{L, R\}$, there exists a polynomial of at most degree 32 realising this transition by Lemma 2. These polynomials are included in our map in order to simulate $\mathcal{A}$. Note, that our simulation is slightly different from [16] as there, in each step, the PRM guessed (and verified) the content of the cell where the head moves in the successive configuration and only correct moves are available due to the state structure. In our model, as there is no state structure, each time a move is simulated, we have to verify that indeed both the state and current cell are correct. The initial value x_0 satisfies

$$x_0 \equiv 1 \mod p_{s+1} \qquad \text{and} \qquad x_0 \equiv 0 \mod p_\ell \text{ if } \ell \neq s+1.$$

The main idea is still the same, if $\mathcal{A}$ is simulated incorrectly, the value x becomes 2 modulo some prime p_ℓ and will remain 2 forever.

It should be also pointed out that simulating behaviour on the border of the tape is easy and results in polynomials of at most degree 32. Moreover, the sequence of configurations $[q_j, 1, \triangleright aw\triangleleft] \to [q_k, 0, \triangleright bw\triangleleft] \to [q_j, 1, \triangleright bw\triangleleft]$ does not require any special considerations as each polynomial contains EQONE_{s+j} ensuring that this sort of loop cannot be used to unfaithfully modify the first letter on the tape.

By induction on the length of the run of LBA $\mathcal{A}$, it is easy to see that $[q_1, 0, \triangleright 0^s \triangleleft] \to^* [q_f, 0, \triangleright 0^s \triangleleft]$ in $\mathcal{A}$ if and only if a residue class r, such that

$$r \equiv 1 \mod p_{s+|Q|} \qquad \text{and} \qquad r \equiv 0 \mod p_\ell \text{ if } \ell \neq s+|Q|,$$

is reachable from x_0 by applying polynomials from the constructed map. To reach 0, we need three additional polynomials: one polynomial to move to a residue class r' such that $r' \equiv 0 \mod p_\ell$ for all $1 \leq \ell \leq m$, and two polynomials to move from the *integer* r' to 0. The first polynomial is $p_{flip,s+|Q|}(p_{eqone,s+|Q|}(x))$ as we assumed that the final state appears only in the configuration $[q_{|Q|}, 0, \triangleright 0^s \triangleleft]$. The latter polynomials are $p_+(x) = x + P$ and $p_-(x) = x - P$.

We have proved the following lemma:

Lemma 3. *The reachability problem is* PSPACE*-hard for polynomial maps over* $\mathbb{Z}[x]$.

We illustrate the simulation of an LBA with polynomials.

Example 4. Let $\mathcal{A}$ be an LBA with a single state, a tape with two cells and a move $\delta(q_1, 0) = (q_1, 1, R)$. We need two primes for the tape and another two for the head. That is, we use primes $7, 11, 13$ and 17. For the sake of readability, we present all the integers modulo $P = 7 \cdot 11 \cdot 13 \cdot 17 = 17017$. The integers r and s representing configurations $[q_1, 1, \triangleright 00 \triangleleft]$ and $[q_1, 2, \triangleright 10 \triangleleft]$ can be solved from the system of congruences

$$\begin{aligned} r &\equiv 0 \mod 7, & s &\equiv 1 \mod 7 \\ r &\equiv 0 \mod 11, & s &\equiv 0 \mod 11, \\ r &\equiv 1 \mod 13, & s &\equiv 0 \mod 13, \\ r &\equiv 0 \mod 17, & s &\equiv 1 \mod 17 \end{aligned}$$

using the Chinese remainder theorem. That is, $r = 3927$ and $s = 715$. The move $\delta(q_1, 0) = (q_1, 1, R)$ is realised by $\text{FLIP}_1 \circ \text{EQZERO}_1 \circ \text{FLIP}_4 \circ \text{FLIP}_3 \circ \text{EQONE}_3$. By Lemma 2, EQONE_3 is realised by a quadratic polynomial $a'_2x^2 + a'_1x + a'_0$ with coefficients satisfying the congruences

$$\begin{aligned} a'_2 &\equiv 0 \mod 7, & a'_1 &\equiv 1 \mod 7, & a'_0 &\equiv 0 \mod 7, \\ a'_2 &\equiv 0 \mod 11, & a'_1 &\equiv 1 \mod 11, & a'_0 &\equiv 0 \mod 11, \\ a'_2 &\equiv 1 \mod 13, & a'_1 &\equiv -2 \mod 13, & a'_0 &\equiv 2 \mod 13, \\ a'_2 &\equiv 0 \mod 17, & a'_1 &\equiv 1 \mod 17, & a'_0 &\equiv 0 \mod 17. \end{aligned}$$

Solving these systems using the Chinese remainder theorem, we see that $p_{eqone,3}(x) = 3927x^2 + 5237x + 7854$. The other polynomials are solved from similar systems of congruences and are

$$\begin{aligned} p_{flip,3} &= 14399x^2 + 11782x + 3927, & p_{flip,4} &= 12012x^2 + 6007x + 8008, \\ p_{eqzero,1} &= 7293x^2 + 2432x, & p_{flip,1} &= 14586x^2 + x + 9724. \end{aligned}$$

Finally, the composition of the polynomials is $p(x) = 11968x^4 + 8041x^3 + 9207x^2 + 11056x + 8569$. It can be easily verified that $p(x)$ simulates the move $\delta(q_1, 0) = (q_1, 1, R)$ from the configuration $[q_1, 1, \triangleright 00 \triangleleft]$ correctly, i.e., $p(r) = s$.

Note, that in the previous example, the polynomial composed of five quadratic polynomials is not of degree 32 but only of degree four. This is the case in general due to the coefficients of the polynomials satisfying the particular congruences. For example, consider the composition of two quadratic polynomials $p_{u,k}(x)$ and $p_{v,\ell}(x)$, where $u, v \in \{\mathit{flip}, \mathit{eqone}, \mathit{eqzero}\}$, $k, \ell \in \{1, \ldots, m\}$ and moreover, they have different index, $k \neq \ell$. Let α, β be the coefficients of the highest power. They can be expressed as integers

$$\alpha = x \cdot \prod_{\substack{j=1,\\ j\neq k}}^{m} p_j \qquad \text{and} \qquad \beta = y \cdot \prod_{\substack{j=1,\\ j\neq \ell}}^{m} p_j,$$

where $x, y \in \mathbb{Z}$ and $k, \ell \in \{1, \ldots, m\}$ and $k \neq \ell$. Now, in the composition of $p_{u,k}(x)$ and $p_{v,\ell}(x)$, the coefficient of x^4 is $\alpha \cdot \beta$, which is

$$\alpha \cdot \beta = x \cdot \prod_{\substack{j=1,\\ j\neq k}}^{m} p_j \cdot y \cdot \prod_{\substack{j=1,\\ j\neq \ell}}^{m} p_j = xy \cdot \prod_{\substack{j=1,\\ j\neq k,\ell}}^{m} p_j \cdot \prod_{j=1}^{m} p_j.$$

The coefficient is divisible by P and, thus, can be replaced by 0 in our considerations. Moreover, a similar calculation shows that the coefficient of x^3 is also divisible by P while the coefficient of x^2 is divisible by $\frac{P}{p_k p_\ell}$. In order to show that this holds in general, that is, when composing four or five polynomials, let us denote $p_{u,i}$, where $u \in \{\mathit{flip}, \mathit{eqone}, \mathit{eqzero}\}$ and $i \in \{1, \ldots, m\}$. Note that that for each polynomial, the coefficient of quadratic and constant terms are divisible by $\frac{P}{p_y}$, where y is the respective index. Let us also fix the order of compositions. We write compositions in the form $(p_{\mathit{flip},k} \circ p_{u,k} \circ p_{\mathit{flip},j} \circ p_{\mathit{flip},i} \circ p_{\mathit{eqone},i})(x)$, where $u \in \{\mathit{eqone}, \mathit{eqzero}\}$, indexes $i, j, k \in \{1, \ldots, m\}$. The order can be interpreted as first verifying that the automaton is in the correct state and position on the tape, $p_{\mathit{eqone},i}$ then updating the state and position on the tape using the next two polynomials, $p_{\mathit{flip},j}, p_{\mathit{flip},i}$. Finally, verifying that the content of the tape was correct using $p_{\mathit{flip},k}$ and rewriting it using $p_{\mathit{flip},k}$. Note that, the outermost polynomial is not present if the simulated move does not rewrite the content of the tape. Also note that as described before, the indexes i, j, k are not arbitrary but here we omit the details for the sake of readability.

Since composition of polynomials is associative, we consider the composition $(p_{\mathit{flip},k} \circ (p_{u,k} \circ (p_{\mathit{flip},j} \circ p_{\mathit{flip},i})) \circ p_{\mathit{eqone},i})(x)$. By the previous observation, the polynomial resulting from the composition $(p_{u,k} \circ (p_{\mathit{flip},j} \circ p_{\mathit{flip},i}))(x)$ is quadratic with the coefficient of the quadratic term being divisible by $\frac{P}{p_k p_j p_i}$. Let us denote it by $b_2x^2 + b_1x + b_0$. Let us then compose $p_{\mathit{eqone},i}(x) = a_2x^2 + a_1x + a_0$ with the above polynomial. That is, we compute

$$b_2(a_2x^2 + a_1x + a_0)^2 + b_1(a_2x^2 + a_1x + a_0) + b_0 = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0.$$

The resulting polynomial is quartic and the coefficients c_4 and c_3 are divisible by $\frac{P}{p_i}$. Indeed, $c_4 = b_2a_2$ and $c_3 = 2b_2a_2a_1$ and hence both are divisible by $\frac{P}{p_i}$. Let us denote $p_{\mathit{flip},k} = d_2x^2 + d_1x + d_0$, where d_2 is divisible by $\frac{P}{p_k}$. The final composition is

$$d_2(c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)^2 + d_1(c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0) + d_0.$$

Observe now that the coefficients of terms x^8, x^7, x^6 and x^5 are $c_4^2d_2$, $2c_3c_4d_2$, $(c_3^2d_2 + 2c_2c_4d_2)$ and $(2c_2c_3d_2 + 2c_1c_4d_2)$, respectively. All four coefficients are divisible by P as both c_4d_2 and c_3d_2 are divisible by P.

Hence for the polynomial $p(x)$ simulating a transition of $\mathcal{A}$, all the coefficients of degrees higher than four are divisible by P. Thus we have our claimed lower bound.

Lemma 5. *The reachability problem is* PSPACE*-hard for polynomial maps over* $\mathsf{Quart}_{\mathbb{Z}}[x]$.

To prove that the reachability problem is PSPACE-complete for general one-dimensional maps, it remains to prove that the problem can be solved in PSPACE.

Lemma 6. *The reachability problem for polynomial maps over* $\mathbb{Z}[x]$ *is in* PSPACE.

Proof. Consider the map as a PRM with a single state and where the transitions are labelled by the polynomials of the map. The reachability problem for PRM can be solved in PSPACE and thus also the reachability problem for polynomial maps is in PSPACE. □

Combining lemmas 5 and 6, we have our main result of the section.

Theorem 7. *The reachability problem for polynomial maps is* PSPACE*-complete.*

Next, we extend the previous results by considering polynomials over the field $\mathbb{Q}$. Additionally, we modify the polynomials to ensure that the image is always in $(0, 1]$.

Let $p(x)$ be a polynomial from our map. Then the corresponding map of rational functions $\mathcal{Q}$ has a function $p'(x) = \frac{1}{p(\frac{1}{x})}$. It is easy to see that in fact p' is of the form $\frac{r(x)}{q(x)}$ for some $r(x), q(x) \in \mathbb{Z}[x]$. We can inherit the lower bound for the reachability for rational functions from Lemma 3.

Corollary 8. *The reachability problem for rational function iteration is* PSPACE*-hard, when the rational functions are of form* $\frac{r(x)}{q(x)} : [0, 1] \to [0, 1]$ *over polynomial ring* $\mathbb{Q}[x]$.

4. Reachability in maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^3$ and $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$

In this section, we consider the reachability problem in multidimensional maps. We will show undecidability for $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^3$ and PSPACE-hardness for $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ when the coefficients of the functions are integers. That is, in the three-dimensional variant, we are investigating functions of the form

$$\begin{cases} x_1 := a_1 x_1 + a_0 \\ x_2 := a'_1 x_2 + a'_0 \\ x_3 := a''_1 x_3 + a''_0 \end{cases},$$

where $a_i, a'_i, a''_i \in \mathbb{Z}$, and show that the reachability problem is undecidable by encoding the PCP in the first two dimensions and using the third dimension to make sure that only one word is constructed. It is important to note that the reachability problem in maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^3$ is undecidable only in the case where affine functions can be of the form $\pm x + a_0$. As we will prove in the following section, the reachability problem is decidable even for polynomials of any degree, as long as none is of the form $\pm x + a_0$, i.e., for maps over $\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ for any n. Finally, we will consider two-dimensional maps, where the polynomials are affine and show that the reachability problem is PSPACE-hard.

Note, that the undecidability of the reachability problem for three-dimensional polynomial maps holds as it is straightforward to simulate a two-dimensional PRM with affine updates with undecidable reachability problem [35] and use the third dimension to simulate state transitions using the construction of the previous section.

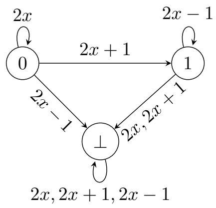

Figure 3: An illustration of the behaviour of the register. State $\perp$ corresponds to register value being other than 0 or 1.

Lemma 9. *The reachability problem for multidimensional polynomial iteration is undecidable for three-dimensional polynomials.*

However, we prove a stronger result using a different reduction. Namely, that the reachability problem is undecidable already for $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^3$.

Theorem 10. *The reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^3$ is undecidable with at least $n_p + 2$ affine functions over $\mathbb{Z}$. (Currently, $n_p = 5$.)*

Proof. Let $\Sigma = \{a, b\}$ be a binary alphabet and let $P = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\} \subseteq \Sigma^* \times \Sigma^*$ be an instance of the PCP. We define a set of three-dimensional affine functions to simulate the PCP. We show that $(0, 0, 1)$ is reachable from $(0, 0, 0)$ if and only if the PCP has a solution. We define sets consisting of three-dimensional affine functions for every pair $(u_i, v_i) \in P$, where $1 \leq i \leq n$.

$$
\begin{aligned}
F_1 &= \{(3^{|u_i|}x_1 + \sigma(u_i), 3^{|v_i|}x_2 + \sigma(v_i), 2x_3) \mid 1 \leq i \leq n\},\\
F_2 &= \{(3^{|u_i|}x_1 + \sigma(u_i), 3^{|v_i|}x_2 + \sigma(v_i), 2x_3 + 1) \mid 1 \leq i \leq n\},\\
F_3 &= \{(x_1 - 1, x_2 - 1, 2x_3 - 1)\},
\end{aligned}
$$

where σ is the embedding of Lemma 1, i.e., ternary representation of a binary word.

Consider the two configurations $(0, 0, 0)$ and $(0, 0, 1)$. We prove the undecidability of the reachability problem in affine maps by showing that the configuration $(0, 0, 1)$ is reachable from the initial configuration $(0, 0, 0)$ by applying affine functions in $F = F_1 \cup F_2 \cup F_3$ if and only if the PCP instance P has a solution.

First, we prove that a configuration $(x, y, 1)$, for some $x, y \in \mathbb{Z}$, is reachable from $(0, 0, 0)$ if and only if first functions from F_1 are applied, then a function from F_2, and finally functions from F_3 are applied. Observe first that affine functions in the third components in the functions from F_1, F_2 and F_3 have fixed points 0, -1 and 1, respectively. Additionally, if x is not a fixed point of a function $p(x)$ from some F_i, then $|p(x)| > |x|$. Now it is easy to see that applying the functions in the claimed order results in third component being 1. Indeed, first 0 is fixed by applying functions from F_1, then it is mapped to $2 \cdot 0 + 1 = 1$ by a function from F_2, and finally is again fixed by functions from F_3. If the functions are applied in a different order, then, as the absolute value does not decrease, value 1 cannot be reached. This is illustrated in Figure 3.

Now we are ready to prove that we can reach both zeros in the first and second dimensions if and only if the PCP has a solution. It is possible to construct an identical pair of words by concatenating the pairs in the PCP instance P if and only if the PCP has a solution. In other words, we can reach a configuration $(y, y, 1)$ for some $y \in \mathbb{N}$ by applying the functions in F_1 and F_2 if and only if we have a solution for the PCP instance P. Then, it is easy to see that the target configuration $(0, 0, 1)$ is reachable by applying the only affine function in F_3 if and only if the PCP has a solution. Recall that the PCP is undecidable for $n_p = 5$ pairs [32]. Note that $|F| = 2n_p + 1$. To achieve the claimed size of $n_p + 2$, we observe that since a function from F_2 is applied only once, we can consider a set

$$F_2' = \{(3^{|u_i|} \cdot x_1 + \sigma(u_i), 3^{|v_i|} \cdot x_2 + \sigma(v_i), 2 \cdot x_3 + 1)\},$$

for some $1 \leq i \leq n$. Indeed, this intuitively corresponds to assuming without loss of generality that the final pair in a potential solution is (u_i, v_i). It is easy to see that the reachability problem is undecidable for $F_1 \cup F_2' \cup F_3$ and there are $n_p + 2$ functions in the set. $\square$

In the previous proof, a state structure was simulated by affine functions. It is possible to generalise the construction of the affine polynomials enforcing a richer state structure. A construction for arbitrary state structure is proven next. We use this result in upcoming results of the paper to embed a state structure of an automaton into affine functions. It will allow us to prove PSPACE-hardness of the reachability problem and undecidability of the universality problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$.

Lemma 11. *Let $G = (V, E)$ be a directed graph with vertices $v_0, \ldots, v_{m-1}$. There exists a set of affine functions F such that for an edge $(v_i, v_j) \in E$, where $0 \leq i, j \leq m - 1$, there exists a unique $f_{ij} \in F$ such that $f_{ij}(i) = j$ and $f_{ij}(x) \notin [0, m-1]$ for all $x \neq i$.*

Proof. Let $G = (V, E)$ be a graph with $V = \{v_0, \ldots, v_{m-1}\}$. For each edge of G, we add an affine polynomial of the form $m \cdot x + b$, for some $b \in \mathbb{Z}$, to the map. The idea is for integers in $[0, m-1]$ to represent each vertex and with the correct affine function, the value of the register changes according to the graph. In the event of wrong affine function being chosen, the coefficient m ensures that the resulting value of the register will be either larger than m or less than zero. Further, in both cases, none of the subsequent functions result in a value in $[0, m-1]$.

Let us formally define the affine polynomials. Let $(v_i, v_j) \in E$ be an edge from v_i to v_j (possibly $i = j$), the corresponding affine function $f_{ij}(x)$ is such that $f_{ij}(i) = j$, that is, $f_{ij}(x) = m \cdot x + b = j$, where $b = j - m \cdot i \in [-m^2 + m, m - 1]$. It is easy to see that $f_{ij}(i') \in \mathbb{Z} \setminus [0, m-1]$ for $i' \neq i$. Indeed, $f_{ij}(i') = m \cdot i' + j - m \cdot i = m(i' - i) + j$ and if $i' > i$, then $m(i' - i) + j > m$, or if $i' < i$, then $m(i' - i) + j < 0$. Note that the previous observation implies that there is no sequence of affine functions $f_{i_1 j_1}, \ldots, f_{i_k j_k} \in F$, where $i_1 \neq i$, such that $f_{i_k j_k}(f_{i_{k-1} j_{k-1}}(\cdots (f_{i_1 j_1}(i)) \cdots) = j$. $\square$

Example 12. We illustrate Lemma 11. Let $G = (V, E)$, where $V = \{v_0, v_1, v_2\}$ and $E = \{(v_0, v_0), (v_0, v_1), (v_1, v_1), (v_1, v_2), (v_2, v_2), (v_2, v_0)\}$. The corresponding set of affine functions is $\{3x - 6, 3x - 4, 3x - 2, 3x - 1, 3x, 3x + 1\}$ and the state structure they define is depicted in Figure 4.

Observe that Lemma 11 gives us a correspondence between an edge in a graph and an affine function. It is easy to see that there is a one-to-one correspondence between a path

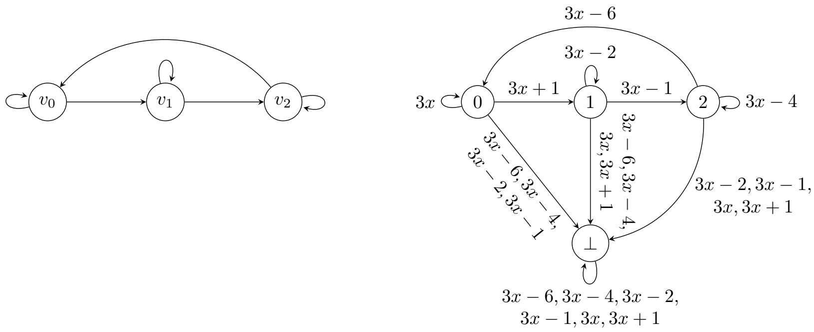

Figure 4: Graph G (left) and an illustration of the behaviour of the register in the corresponding affine map (right). State $\perp$ corresponds to register value being other than 0, 1, 2.

from vertex v_i to vertex v_j in the graph and a sequence of affine functions transforming i into j. We can use Lemma 11 to embed a state structure into a one-dimensional affine map. In the next theorem, we embed a one-dimensional ARM of [24] into a two-dimensional map, where the functions are affine.

Theorem 13. *The reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ is* PSPACE*-hard.*

Proof. Let $\mathcal{R} = (Q, \Delta)$ be a one-dimensional ARM with PSPACE-hard reachability problem constructed in [24]. Denote $Q = \{q_0, \ldots, q_{m-1}\}$. For each transition $(q_i, p(x), q_j)$ of $\mathcal{R}$, we add two-dimensional function $(p(x), m \cdot x + j - m \cdot i)$ to the map. It is clear that $(0, k)$ is reachable from $(0, \ell)$ if and only if $[q_\ell, 0] \to^*_{\mathcal{R}} [q_k, 0]$. That is, the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ is PSPACE-hard. ☐

There are two main open problems for dimension two. Namely, are the reachability problems for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ or $\mathbb{Z}[\vec{x}]^2$ decidable?

5. Reachability in maps without additive updates

In this section, we consider a restricted class of maps over $\mathsf{Aff}_{\mathbb{Z}}[x]$, in the sense that every affine function in the map is not of the form $\pm x + a_0$. It is easy to see that the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[x]$ is NP-hard as we can easily reduce the *subset sum problem* (SSP) [26] to the reachability problem in maps. The NP-hardness proof relies on the use of polynomials of the form $x + a_0$ that correspond to integers in the SSP. Then, we can easily reduce the SSP with a target integer s to the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[x]$ with a target s. If we further restrict ourselves to maps where *all* polynomials are of the form $\pm x + a_0$, then the reachability problem is NP-complete [10, 19]. However, we do not have a tight complexity bound for the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[x]$ to the best of our knowledge. The best upper bound we know of is the PSPACE upper bound given by Finkel et al. [16] following from the PSPACE-completeness of the reachability problem for one-dimensional PRMs.

We consider a dual setting, where all polynomials of the form $\pm x + a_0$ are excluded, i.e., maps over $\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$. We will prove that the reachability problem for maps over $\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is in PSPACE for any dimension n. Then, we establish the PSPACE-completeness for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$, when n is not fixed, by proving the hardness via a reduction of the reachability problem for LBA. Note that Theorem 13 does not yield a lower bound for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ as the ARM of [24] relies on affine updates of the form $\pm x + a_0$.

Let us first prove that the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ remains NP-hard. That is, we give an alternative proof for NP-hardness of the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[x]$ that does not rely on additive updates.

Lemma 14. *The reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ is* NP*-hard.*

Proof. We shall reduce the SSP to the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$. Let (S, s) be an instance of the SSP, where $S = \{s_1, s_2, \ldots, s_k\}$ is a set of positive integers and s is a target integer. Recall that the SSP is a well-known NP-hard problem, where the task is to decide whether there exists any subset of S whose elements sum up to the target integer s [26].

Let $n > \max(S) \cdot k$ be an integer with representation of size polynomial in the input. We define the set of affine functions as follows:

$$F = \{n \cdot x + n^{i-1} \cdot s_i \mid 1 \le i \le k\} \cup \{n \cdot x\}.$$

Now we prove that the map reaches $n^{k-1} \cdot s$ from 0 if and only if the instance (S, s) has a solution. Assume first that there exists a subset $S' \subseteq S$ such that $\sum_{x \in S'} x = s$. Let us go through S element by element, starting with s_1. We apply $n \cdot x + n^{i-1} \cdot s_i$ or $n \cdot x$ depending on whether $s_i \in S'$ or not. Indeed, let $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$ be the set of indexes in S' in increasing order. Now the composition of functions is

$$\begin{aligned}
&n^{k-i_m}(n^{i_m - i_{m-1}}(\cdots(n^{i_2-i_1}(n^{i_1} \cdot 0 + n^{i_1-1} \cdot s_{i_1}) + n^{i_2-1} \cdot s_{i_2})\cdots) + n^{i_{m-1}-1} s_{i_m})\\
&\qquad = n^{k-i_m}(n^{i_m-i_{m-1}}(\cdots(n^{i_2-1} \cdot (s_{i_1} + s_{i_2}))\cdots) + n^{i_{m-1}-1} s_{i_m})\\
&\qquad = \cdots = n^{k-i_m}(n^{i_m-1}(s_{i_1} + \cdots + s_{i_m}))\\
&\qquad = n^{k-1}(s_{i_1} + \cdots + s_{i_m}) = n^{k-1} s.
\end{aligned}$$

Assume then that (S, s) has no solution and assume towards a contradiction that the map reaches the integer $n^{k-1} \cdot s$. Let $f_1(x), f_2(x), \ldots, f_m(x)$ be a sequence of affine functions applied to 0 that results in $n^{k-1} \cdot s$. We can assume that $f_1(x) \neq n \cdot x$. Indeed, otherwise $f_1(0) = n \cdot 0 = 0$ and we can consider the sequence $f_2(x), \ldots, f_m(x)$ instead. Observe that $m < k$, since $f_1(0) = n^{i-1} \cdot s_i$ for some i, and each subsequent function multiplies this value by n. Hence, after k applications of the functions, the value is at least $n^k \cdot s_i$. Further notice that at every intermediate step, the value is of the form $n^\ell \cdot y$ for some $\ell \in \{0, \ldots, k-1\}$ and $0 < y < n$. Indeed, if after applying $f_j(x)$, the result is $n^{\ell_1} \cdot y_1 + n^{\ell_2} \cdot y_2$, where $\ell_2 < \ell_1$ and $0 < y_2 < n$, then after applying the next function $f_{j+1}(x)$, the result is $n^{\ell_1+1} \cdot y_1 + n^{\ell_2+1} \cdot y_2 + n^{i-1} \cdot s_i$ for some i, or $n^{\ell_1+1} \cdot y_1 + n^{\ell_2+1} \cdot y_2$. In both cases the coefficients of the terms n^{ℓ_1+1} and n^{ℓ_2+1} are non-zero. Following this reasoning, after applying the functions $f_{j+2}(x), \ldots, f_m(x)$, the integer reached will have non-zero coefficients for terms $n^{\ell_1+m-j-1}$ and $n^{\ell_2+m-j-1}$. This is a contradiction as we assumed that after applying the sequence of affine functions, the reached integer is $n^{k-1} \cdot s$. It follows that a function $n \cdot x + n^{i-1} \cdot s_i$ appears in the sequence at most once.

Now we can extract s_i's from the sequence and see that they form a solution for the instance (S, s) of the SSP which is a contradiction.

Since the integer $n^{k-1} \cdot s$ is reachable in maps over $\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ if and only if the SSP instance (S, s) has a solution, we can conclude that the reachability problem for $\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ is also NP-hard. □

Like the case of maps over $\mathsf{Aff}_{\mathbb{Z}}[x]$, we know that the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ is decidable in PSPACE and is NP-hard. However, unlike the general case of affine maps where the problem becomes undecidable in higher dimensions, the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ stays in PSPACE for any dimension. The reachability problem remains in PSPACE even if we do not impose a limit on the degree of polynomials, that is, for $\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$.

Theorem 15. *The reachability problem for maps over $\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is decidable in* PSPACE *for any $n \geq 1$.*

Proof. Consider first $n = 1$. We follow the reasoning of [16]. In Lemma 3 of [16], it was proven that there exists a bound $b \in \mathbb{N}$ such that every polynomial $p(x) \in \mathbb{Z}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$ is monotonically increasing or decreasing in $\mathbb{Z} \setminus [-b, b]$ and, moreover, $|p(y)| \geq 2|y|$ for all $y \in \mathbb{Z} \setminus [-b, b]$. Note that the size of the representation of b is polynomial in size of the representation of the input. Namely, $b = d(a + 2)$, where d is the maximal degree of polynomials in the map and a the largest absolute value of the coefficients of polynomials in the map. The monotonicity follows from the fact that the first derivative of $p(x)$ has no roots outside $[-b, b]$. The inequality $|p(y)| \geq 2|y|$ holds for all $y \in \mathbb{Z} \setminus [-b, b]$ because $p(y)$ is monotonic and does not intersect lines $\pm 2y$ when y is outside $[-b, b]$. The details can be found in the Appendix of [16].

Let z be the target integer. If $|z| \leq b$, we can decide whether the integer z is reachable in PSPACE by applying the given functions since we can store the current value and the computation path in space polynomial in b which again has representation of polynomial size in the size of the input. If z is outside the interval $[-b, b]$, due to the properties of polynomials in $\mathbb{Z}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$, do not need to consider the integers outside the interval $[-z, z]$. Namely, because for every $p(x) \in \mathbb{Z}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$, $|p(y)| \geq |y|$ for all $y \in \mathbb{Z} \setminus [-b, b]$, once the map reaches an integer outside $[-z, z]$, it can never reach an integer in $[-z, z]$. That is, still only polynomial space in size of the input is required.

The PSPACE upper bound holds even if we consider n-dimensional case since we can maintain the information of the current computation for each dimension in space polynomial in the input. □

Corollary 16. *The reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is decidable in* PSPACE *for any $n \geq 1$.*

Next we show that by considering a larger domain, the undecidability for three-dimensional maps remains, even without additive updates. That is, we prove that the reachability problem is undecidable for maps over $\mathsf{Aff}_{\mathbb{Q}}[\vec{x}]^3 \setminus \mathsf{Add}_{\mathbb{Q}}$, i.e., when the functions are of the form $a_1 x + a_0$, where $a_1, a_0 \in \mathbb{Q}$ and $a_1 \neq \pm 1$.

Theorem 17. *The reachability problem for rational maps over $\mathsf{Aff}_{\mathbb{Q}}[\vec{x}]^3 \setminus \mathsf{Add}_{\mathbb{Q}}$ is undecidable with at least $2n_p + 1$ affine functions over $\mathbb{Q}$. (Currently, $n_p = 5$.)*

<table>
<tr><th>type / dim.</th><th>$\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$</th><th>$\mathsf{Aff}_{\mathbb{Z}}[x]$</th><th>$\mathsf{Aff}_{\mathbb{Q}}[x] \setminus \mathsf{Add}_{\mathbb{Q}}$</th></tr>
<tr><td>1</td><td rowspan="4">NP-h./PSPACE</td><td>NP-h. [19]/PSPACE [16]</td><td rowspan="2">NP-h./?</td></tr>
<tr><td>2</td><td>NP-h. [19]/?</td></tr>
<tr><td>3</td><td rowspan="3">undecid.</td><td rowspan="3">undecid.</td></tr>
<tr><td>$\vdots$</td></tr>
<tr><td>n</td><td>PSPACE-c.</td></tr>
</table>

Table 4: The differences of complexity results for affine polynomials depending on the underlying ring with and without additive updates. The results of this section are highlighted in grey.

Proof. Again, we reduce the PCP to show the undecidability of the problem but in a slightly different way.

Let P be an instance of the PCP with n elements over a binary alphabet. Let us define sets consisting of three-dimensional affine functions for every pair $(u_i, v_i) \in P$, where $1 \leq i \leq n$:

1. $(3^{|u_i|} \cdot x_1 + \sigma(u_i), (n+1) \cdot x_2 + i, 2 \cdot x_3) \in F_1$ for all $1 \leq i \leq n$,
2. $(3^{|u_i|} \cdot x_1 + \sigma(u_i), (n+1) \cdot x_2 + i, 2 \cdot x_3 + 1) \in F_2$ for some $1 \leq i \leq n$, and
3. $\left(\frac{1}{3^{|v_i|}} \cdot (x_1 - \sigma(v_i)), \frac{1}{n+1} \cdot (x_2 - i), 2 \cdot x_3 - 1\right) \in F_3$ for all $1 \leq i \leq n$,

where σ is the embedding of Lemma 1. Observe, that F_2 consists of a single affine function. We implicitly use the same approach as in Theorem 10, to assume without loss of generality that the pair (u_i, v_i) is used last in a solution of the PCP.

Observe that none of the affine functions is of the form $\pm x + a_0$, in contrast to the functions in the proof of Theorem 10. The main idea is that we first construct a word $u' = u_{i_1} u_{i_2} \cdots u_{i_{k-1}}$, where $1 \leq i_j \leq n$ for all $1 \leq j \leq k-1$, in the first dimension by applying affine functions from F_1, corresponding to the words from u_1 to u_n. Then, as in the proof of Theorem 10, we apply a function from F_2 once, resulting in $u = u' u_{i_k}$ in the first component. Next, we cancel out the constructed word by applying affine functions that are multiplicative inverses of affine functions corresponding to the words from v_1 to v_n. Note that $u_{i_1} u_{i_2} \cdots u_{i_k} = v_{i_1} v_{i_2} \cdots v_{i_k}$ should hold due to the second dimension where we force the sequences of indices of words to be matched. Hence, we see that the configuration $(0,0,1)$ can be reached from the initial configuration $(0,0,0)$ if and only if the PCP instance P has a solution. $\square$

Now we prove the PSPACE-hardness of the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ by reducing the reachability problem of LBA to it. Note that the dimension n is not fixed.

Lemma 18. *The reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is* PSPACE*-hard.*

Proof. Let $\mathcal{A} = (Q, \Gamma, \delta)$, where $Q = \{q_0, \ldots, q_{m-1}\}$, be an LBA with PSPACE-hard reachability problem. Recall that in the reachability problem for LBAs, we are asked whether $(q_0, \triangleright 0^s \triangleleft, 0) \to_{\mathcal{A}}^* (q_{m-1}, \triangleright 0^s \triangleleft, 0)$ holds. We reduce this reachability problem to the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^{s+1} \setminus \mathsf{Add}_{\mathbb{Z}}$. Let us define the set $F_{\mathcal{A}}$ of affine functions that simulate the computation of $\mathcal{A}$.

The main idea is that we store the tape content of the LBA $\mathcal{A}$ in the first s dimensions of the affine map. Then, we maintain the information of the current state and the head position in the final dimension using the construction of Lemma 11.

Let $(q_j, \triangleright w \triangleleft, i)$ be the current configuration of $\mathcal{A}$, where $0 \leq j \leq m-1$ and $1 \leq i \leq |w|$. Denote $w = w_1 w_2 \cdots w_s \in \{0,1\}^s$. The corresponding register value in the affine map is as follows:

$$(\underbrace{w_1, w_2, \ldots, w_s}_{s}, z),$$

where z corresponds to the head being in state q_j in position i of the tape.

First, let us construct a state structure incorporating information on both state and position of the head of $\mathcal{A}$. That is, we construct a graph $G_{\mathcal{A}}$ with vertices $(q,i) \in Q \times [1,s]$ and with edges $((q,i),(q',i-1))$ if there is a transition $(q,a,q',b,L) \in \delta$, and $((q,i),(q',i+1))$ if there is a transition $(q,a,q',b,R) \in \delta$, for any $a,b \in \Gamma$. Note that $G_{\mathcal{A}}$ does not take the tape content into account. By Lemma 11, the graph $G_{\mathcal{A}}$ can be simulated by affine functions. We omit details on the behaviour of $\mathcal{A}$ on the endmarkers $\triangleright$ and $\triangleleft$. These transitions can be easily hardcoded into the graph.

Then, to simulate rewriting of the tape, we consider a graph G_Γ with two vertices, 0 and 1, and edges are $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Intuitively, vertex 0 corresponds to the symbol being 0 and edge $(0,0)$ corresponds to the head not rewriting the symbol, while $(0,1)$ means that the symbol was rewritten as 1. By Lemma 11, there exists a set of affine functions simulating this behaviour.

We are now ready to combine the two sets of affine functions in order to simulate $\mathcal{A}$. Let us consider transition $(q_{j_1}, 0, q_{j_2}, 1, L) \in \delta$ and the current head position is i where $1 \leq i \leq s$. This transition switches the state from q_{j_1} to q_{j_2} if the tape has 0 in the current head position while writing 1 and moving the head position to the left. This implies that in $G_{\mathcal{A}}$ constructed previously, we need to take the edge $((q_{j_1}, i), (q_{j_2}, i-1))$ to successfully perform the transition. Further, in the ith dimension, we apply edge $(0,1)$ of G_Γ. Note that we apply the identity function in the other dimensions unless explicitly mentioned. The affine function corresponding to $(q_{j_1}, 0, q_{j_2}, 1, L)$ is

$$(x, \ldots, x, 2x+1, x, \ldots, x, a \cdot x + b),$$

where $a \cdot x + b$ corresponds to the edge $((q_{j_1}, i), (q_{j_2}, i-1))$ in $G_{\mathcal{A}}$, and $2x+1$ is in the ith dimension. The transitions moving the head to right are defined analogously.

Since we have proved that every transition of $\mathcal{A}$ can be simulated by an affine function, it is clear that if $(q_0, \triangleright 0^s \triangleleft, 0) \to_{\mathcal{A}}^* (q_{m-1}, \triangleright 0^s \triangleleft, 0)$ holds in $\mathcal{A}$, then we can reach the register in the affine map corresponding to $(q_{m-1}, \triangleright 0^s \triangleleft, 0)$ from the register corresponding to the initial configuration $(q_0, \triangleright 0^s \triangleleft, 0)$ of $\mathcal{A}$. Therefore, we have proved that the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is PSPACE-hard for unbounded n. $\square$

Based on Corollary 16 and Lemma 18, we have the following main result:

Theorem 19. *If the dimension n is not fixed, the reachability problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is* PSPACE*-complete.*

It is not difficult to see that the PSPACE-completeness holds for multidimensional ARMs and PRMs without additive updates as well.

Corollary 20. *If the dimension n is not fixed, the reachability problem for n-ARMs and n-PRMs, where the update polynomials are not of the form $\pm x + a_0$, is* PSPACE*-complete.*

Corollary 21. *If the dimension n is not fixed, the reachability problem for maps over $\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}$ is* PSPACE*-complete.*

We consider a sort of dual of Lemma 11 by investigating PRMs, where the state structure is induced by affine functions. Let $F \subseteq (\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}) \times \mathbb{Z}[x]$ and $\vec{x}_f$ be the target value of the register respectively. We first consider the first component and proceed as in the proof of Theorem 15. That is, we find the interval $[-b, b]$ where the affine polynomials have non-trivial behaviour. Recall that the representation of b is of polynomial size. It is defined as $b = \max(a + 2, |x_f|)$, where a is the largest absolute value of the coefficients of polynomials and x_f is the first component of the target. We construct a PRM with $2b + 2$ states where the transitions are defined by the affine functions. More precisely, the $2b + 1$ states correspond to integers in the interval $[-b, b]$ and one vertex $\perp$ corresponds to integers outside the interval. As the integers are encoded in binary, there are exponentially many integers in the interval. Let (f_1, f_2) be a function in the map. For each $x \in [-b, b]$, we add a transition $(x, f_2, f_1(x))$ if $f_1(x) \in [-b, b]$ and $(x, f_2, \perp)$ otherwise. We also add transition $(\perp, f_2, \perp)$. There are exponentially many states in the size of the input and hence the PRM requires exponential space. Since the reachability problem for PRMs is in PSPACE, the reachability problem for maps over $(\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}) \times \mathbb{Z}[x]$ is in EXPSPACE.

Theorem 22. *The reachability problem for maps over $(\mathsf{Aff}_{\mathbb{Z}}[x] \setminus \mathsf{Add}_{\mathbb{Z}}) \times \mathbb{Z}[x]$ is in* EXPSPACE*.*

We can present a stronger result. Namely, if we consider maps in higher dimensions with the same restrictions that only one component is $\mathbb{Z}[x]$ and the other components do not have polynomials in $\mathsf{Add}_{\mathbb{Z}}$, then the reachability problem is in EXPSPACE. That is, for maps $(\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}) \times \mathbb{Z}[x]$. Indeed, using the same approach, we construct a PRM with states consisting of n components each corresponding to $[-b_i, b_i]$, where $i \in \{1, \ldots, n\}$ and each bound b_i is defined as the before. Let us denote this set by $\mathcal{B}$. Note that the number of states is still exponential in the size of the input. Let $f_1, \ldots, f_n, p$ be a function in the map. For every $(x_1, \ldots, x_n) \in \mathcal{B}$, then the PRM has a transition $((x_1, \ldots, x_n), p, (f_1(x_1), \ldots, f_n(x_n)))$ if $(f_1(x_1), \ldots, f_n(x_n)) \in \mathcal{B}$ and $((x_1, \ldots, x_n), p, \perp)$ otherwise. Finally, by considering the reachability for PRMs, we obtain the following theorem.

Theorem 23. *The reachability problem for maps over $(\mathbb{Z}[\vec{x}]^n \setminus \mathsf{Add}_{\mathbb{Z}}) \times \mathbb{Z}[x]$ is in* EXPSPACE*.*

While this result seems rather artificial, it is the only decidability result, that we know of, for multidimensional maps where all components are not restricted to $\mathbb{Z}[x] \setminus \mathsf{Add}_{\mathbb{Z}}$.

6. Maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ as language acceptors

In this section, we extend our models to operate on words. Then it is natural to consider the languages accepted by the maps. In this context, the reachability problems of the previous sections can be seen as language emptiness problems. Indeed, the language accepted by a map is empty if and only if the final configuration is not reachable from the initial configuration. The complementary problem to the emptiness problem is the *universality* problem, where we are asked whether every word is accepted by the computational model. We show that for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ the universality problem is undecidable. This contrasts the known results on the emptiness problem as we showed that the emptiness problem is undecidable for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^3$ and NP-hard for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$. The model is connected to blind counter automata and VAS languages that have been extensively studied in the past [18, 25, 13].

In [4] the authors study the zeroness problem, which is essentially the universality problem we consider in this section. The main difference is that our model is nondeterministic, while in [4] the automaton is deterministic. The authors show that the zeroness problem is Ackermann-complete[3].

Let us define the language acceptors more precisely. Let $G \subseteq (\Sigma \cup \{\varepsilon\}) \times F$, where $F \subseteq \mathbb{Z}[\vec{x}]^n$ is an n-dimensional polynomial map, $\vec{x}_0$ and $\vec{x}_f$ the initial and target vectors, respectively, and Σ is an alphabet. A word $a_{i_1}a_{i_2}\cdots a_{i_k}$, where $a_{i_j} \in \Sigma \cup \{\varepsilon\}$, is accepted by G, i.e., is in $L(G)$, if there exists $\pi = ((a_{i_1}, f_{i_1}), (a_{i_2}, f_{i_2}), \ldots, (a_{i_k}, f_{i_k}))$ such that $f_{i_k}(\cdots f_{i_2}(f_{i_1}(\vec{x}_0))\cdots) = \vec{x}_f$. In the universality problem, given $G \subseteq \Sigma \times F$, we are asked to decide whether $L(G) = \Sigma^*$ or not.

Before proving that the universality problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ is undecidable, we recall the definition of integer weighted automata of [20]. An *integer weighted automaton* $\mathcal{A}^\gamma$ is an NFA with integer weights on the transitions. The automaton accepts word w if on the accepting path reading w, the sum of weights encountered is zero. Note that the automata of [20] were defined without final states, i.e., $F = Q$. The transitions of $\mathcal{A}^\gamma$ are of the form (q_i, a, q_j, z), with which the automaton reads letter $a \in \Sigma$ in state q_i, changes its state to q_j and adds z to the weight, i.e., applies function $f(x) = x + z$. In [20] it was proven that it is undecidable whether $L(\mathcal{A}^\gamma) = \Sigma^*$ for integer weighted automata with at least four states.

Theorem 24. *The universality problem for maps over $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ is undecidable.*

Proof. Let $\mathcal{A}^\gamma$ be an integer weighted automaton over alphabet Σ for which the universality problem is undecidable. Let $m-1$ be the number of states of $\mathcal{A}^\gamma$ and let us enumerate them as $q_0, \ldots, q_{m-2}$ such that q_0 is the initial state. To accept words in a unique state, we introduce a new state q_{m-1} and transition (q_i, a, q_{m-1}, z) for each transition (q_i, a, q_j, z) of $\mathcal{A}^\gamma$. The idea is to encode $\mathcal{A}^\gamma$ into maps in such way that the second dimension is used to simulate the state transitions of the automaton using Lemma 11. Let (q_i, a, q_j, z) be a transition of $\mathcal{A}^\gamma$. The corresponding two-dimensional affine function is $(a, (x_1 + z, m \cdot x_2 + j - m \cdot i))$. Now, a word $w \in \Sigma^*$ is accepted by the map if and only if the register values $(0, m-1)$ are reachable from $(0, 0)$ while reading word w. It is clear that the map accepts w if and only if the automaton accepts w. That is, it is undecidable whether the map accepts every finite word. $\square$

In [20], the constructed integer weighted automaton has alphabet of size $|\Gamma| + |\delta| + 3$, where Γ is the tape alphabet and δ is the transition function of a Turing machine with one-way infinite tape.

[3] Ackermann is a complexity class containing decision problems solvable in time bounded by Ackermann function, which is computable but not primitive-recursive.

We further investigate the properties of the reachability sets of different maps. Let us first define a reachability set. Let $F \subseteq \mathbb{Z}[\vec{x}]^n$ be a map over $\mathbb{Z}[\vec{x}]^n$ and let $\vec{x}_0 \in \mathbb{Z}^n$ be the initial value. The *reachability set* of F is defined iteratively:

$$\begin{aligned}
\mathsf{Reach}_0(F) &= \{\vec{x}_0\},\\
\mathsf{Reach}_i(F) &= \{f(\vec{x}) \mid \vec{x} \in \mathsf{Reach}_{i-1}(F), f \in F\},\\
\mathsf{Reach}(F) &= \bigcup_{i=0}^{\infty} \mathsf{Reach}_i(F).
\end{aligned}$$

Next we show that the intersection non-emptiness problem is undecidable for $\mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ using the PCP. That is, for $F, G \subseteq \mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ whether $\mathsf{Reach}(F) \cap \mathsf{Reach}(G) \neq \emptyset$. The idea is for F (resp. G) to construct ternary representation of words u_i (resp. v_i) in the first dimension and the corresponding indices in the second. Then the intersection is non-empty if and only if it possible to construct the same word, i.e., the PCP has a solution.

Lemma 25. *Let F and G be two-dimensional affine maps. It is undecidable whether the intersection of the respective reachability sets is empty or not.*

Proof. Let $\Sigma = \{a, b\}$ and let $P = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\} \subseteq \Sigma^* \times \Sigma^*$ be an instance of the PCP. We construct two sets consisting of affine maps using the embedding of Lemma 1:

$$\begin{aligned}
F &= \{(3^{|u_i|} \cdot x_1 + \sigma(u_i), (n+1) \cdot x_2 + i) \mid 1 \leq i \leq n\},\\
G &= \{(3^{|v_i|} \cdot x_1 + \sigma(v_i), (n+1) \cdot x_2 + i) \mid 1 \leq i \leq n\}.
\end{aligned}$$

It is clear that $\mathsf{Reach}(F)$ (resp. $\mathsf{Reach}(G)$) consists of ternary representation of words u_i (resp. v_i) in the first dimension and the corresponding indices in the second. Now the intersection of $\mathsf{Reach}(F)$ and $\mathsf{Reach}(G)$ is non-empty if and only if there exists a solution to the PCP instance. $\square$

By modifying the proof idea of the previous lemma, we show that the language intersection problem is also undecidable. The idea is to construct two maps each accepting indices of one of the components of the PCP instance. Then the intersection of the languages is non-empty if and only if there exists a sequence of indices such that both maps store the same ternary representations.

Theorem 26. *Let $F, G \subseteq \Sigma \times \mathsf{Aff}_{\mathbb{Z}}[\vec{x}]^2$ and $\vec{x}_{0_F}$, $\vec{x}_{0_G}$ and $\vec{x}_{f_F}$, $\vec{x}_{f_G}$ be the respective initial and target values. It is undecidable whether the intersection of the respective languages is empty, that is, whether $L(F) \cap L(G) = \emptyset$ holds or not.*

Proof. Let $\Gamma = \{a, b\}$ and let $P = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\} \subseteq \Gamma^* \times \Gamma^*$ be an instance of the PCP and let $\Sigma = \{1, \ldots, n, \#\}$, where # is a fresh symbol. We construct the affine maps using the embedding of Lemma 1:

$$\begin{aligned}
F = \{&(i, (3^{|u_i|} \cdot x_1 + \sigma(u_i), 2x_2)) \mid 1 \leq i \leq n\}\\
&\cup \{\#, (x_1 - 1, 2x_2 + 1)\} \cup \{\#, (x_1 - 1, 2x_2 - 1)\},\\
G = \{&(i, (3^{|v_i|} \cdot x_1 + \sigma(v_i), 2x_2)) \mid 1 \leq i \leq n\}\\
&\cup \{\#, (x_1 - 1, 2x_2 + 1)\} \cup \{\#, (x_1 - 1, 2x_2 - 1)\}.
\end{aligned}$$

Again, the second dimension is used to enforce the correct order of functions. Let $\vec{x}_{0_F} = \vec{x}_{0_G} = (0, 0)$ be the initial value and $\vec{x}_{f_F} = \vec{x}_{f_G} = (0, 1)$ be the target value. It is clear that $L(F)$ (resp. $L(G)$) consists of ternary representation of words u_i (resp. v_i) in the first dimension followed by $\#^*$. Now the intersection of $L(F)$ and $L(G)$ is non-empty if and only if there exists a word $w \in \Sigma^*$ such that $w = w'\#^m$, where $w' \in (\Sigma \setminus \{\#\})^*$ is a sequence of indices of a solution to the PCP instance. □

7. Conclusions

In this paper we studied the reachability problem for polynomial maps. We showed differences in the complexities under various restrictions. Namely, we showed that if the functions in the maps are restricted to affine polynomials, the problem is undecidable already in dimension three. On the other hand, two-dimensional affine maps are PSPACE-hard and no upper bound is known, while one-dimensional maps with quartic functions are PSPACE-complete. Another limitation to the maps leads to decidability in any dimension. If the maps do not contain functions of the form $\pm x + a_0$, then the reachability problem is NP-hard for a fixed dimension and if the dimension is not fixed, the problem is PSPACE-hard. The reachability problem is in PSPACE regardless of the dimension. This does not extend to larger rings as the reachability problem remains undecidable for three-dimensional maps where the coefficients are rational numbers even when functions of the form $\pm x + a_0$ are excluded.

Furthermore, we extended the model to language acceptors and showed that, on top of the above mentioned language emptiness results, the language universality is undecidable starting from dimension two even for affine maps.

There remain several open questions regarding the maps we have considered. Most notably, two-dimensional maps do not have upper bounds and there is a complexity gap for one-dimensional maps with functions of degree at most three. We showed two ways to encode state structure – one based on encoding the finite state space into residue classes and using quartic polynomials to move between residue classes and another based on encoding the finite state space as integers and moving between them using affine functions. It would be interesting to apply the techniques to other computational models. There is also a plethora of language-theoretic questions related to the model such as closure properties of the languages, coverability and separability, as well as characterisations of the languages.

Acknowledgement

The authors would like to thank the anonymous reviewers for their helpful comments and suggestions.

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