A Total Opportunity Cost Matrix Tiebreaker Vogel's Approximation Method for Transportation Problems

SPENCER MADAMEDON

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Abstract

The allocation of resources is one of the most significant challenges met by decision-makers because it affects any company's profitability. One sort of resource allocation problem is transportation problems; in this case, the decision-maker must choose the quantity of goods to be delivered from various sources to various destinations at the lowest possible cost. Therefore, in this research, we have explored the state-of-the-art method Total Opportunity Cost Matrix Vogel's Approximation Method and noticed that it has a drawback as it arbitrarily makes allocations when there are ties in the decision-making process, and we have then developed a novel and effective algorithm after controlling for this limitation. The Total Opportunity Cost Matrix Tiebreaker Vogel's Approximation Method, which is the proposed method, systematically breaks ties at several levels in the iteration process of decision-making and produces an improvement on the state-of-the-art method. Additionally, for continuous cost transportation problems, since it is difficult to have ties, an extension of the proposed method known as Total Opportunity Cost Matrix Tiebreaker Vogel's Approximation Method Threshold uses a percentage threshold to induce ties at the maximum penalty which provides the algorithm with alternative pathways to access more costs that can be considered as the minimum cost during the iteration process, resulting, on average, in a lower initial basic feasible solution. This study compared the performance of the state-of-the-art method with 20,000 simulated balanced transportation problems with real-valued costs and 35 benchmark balanced transportation problems with integer cost values from previously published literature. The results show that, on average, the state-of-the-art method can be improved by about 2% when we take a range of percentage thresholds as the maximum penalty.

Although we are aware of the modest increase in computational complexity of this proposed method (which is not expensive to run), we point out that the quality of the initial basic feasible solution obtained can have a significant impact on business overheads as it is generally believed that the better the initial basic feasible solution obtained, the smaller is the number of iterations required to obtain the optimal solutions saving the company time and money overall. Additionally, apart from the fact that it is simple to understand, this proposed method's best quality is that it can be used with other existing optimisation methods by inducing ties to break ties and can also add another step to methods that break ties, for example using the maximum mean cost to break a tie in the minimum cost for transportation problems.

DECLARATION

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institutes of learning.

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Madamedon S, Correa E, Lisboa P. (2022) Tiebreaker Vogel's Approximation Method, a Systematic Approach to improve the Initial Basic Feasible Solution of Transportation Problems. 12th IEEE Symposium on Computer Applications & Industrial Electronics (ISCAIE 2022).

THESIS ACRONYMS

ASM	Abdul Shakeel Masood
ATM	Allocation Table Method
AVAM	Advanced Vogel's Approximation Method
BCE	Bilqis Chastine Erma
CMS	Column Minimum Subtraction
HMM	Harmonic Mean Method
IASM	Improved Abdul Shakeel Masood
IBFS	Initial Basic Feasible Solution
JHM	Juman and Hoque Method
LCM	Least Cost Method
MDEDM	Maximum Difference Extreme Difference Method
MODI	Modified Distribution Method
NWCM	Northwest Corner Method
RAM	Russell Approximation Method
RCM	Reduced Cost Matrix
RMS	Row Minimum Subtraction

TBVAM	Tiebreaker Vogel's Approximation Method
TDM	Total Difference Method
TDM1	Total Differences Method 1
TOC	Total Opportunity Cost
TOCM-EDM	Total Opportunity Cost Matrix- Extremum Difference Method
TOCM-MEDM	Total Opportunity Cost Matrix-Modified Extremum Difference Method
TOCM-MT	Total Opportunity Cost Matrix-Minimal Total
TOCM-SUM	Total Opportunity Cost Matrix-Sum
TOCM-TBVAM	Total Opportunity Cost Matrix-Tiebreaker Vogel's Approximation Method
TOCM-TBVAM-TH	Total Opportunity Cost Matrix-Tiebreaker Vogel's Approximation Method Threshold
TOCM-VAM	Total Opportunity Cost Matrix-Vogel's Approximation Method
TOCT	Total Opportunity Cost Table
TP	Transportation Problem
VAM	Vogel's Approximation Method

CHAPTER ONE

Research Overview

1.0 Introduction

The first chapter expresses the reasons for conducting the proposed research, beginning with a brief introduction to the research topic, the motivation behind the research, the research questions, the aims, and objectives, and finally, the expected contributions to knowledge.

1.1 Research Overview

Transportation problems have been a well-known problem in operations research since its first mathematical formulation in the 18th century, and it became more prominent during the First World War. At this time in the war, schedules were developed for concentrating troops with the necessary equipment at key depots, which were then quickly dispatched to the designated position, utilising efficient use of optimisation in transportation problems.

Transportation has a major impact on both achieving the economic and social goals of an organisation. For example, the transportation of raw materials or goods from the warehouse to the consumer may be affected by the time taken to deliver the goods and, inevitably, the cost of the unit goods. Other challenges, such as route choice, traffic conditions, and vehicle types, may impede the movement of goods from suppliers to consumers, raising the unit cost of transportation.

In the last two decades, there has been a significant contribution toward the development of a new generation of transportation systems that fit the value proposed to each company, focusing on distinct strengths such as quality, speed, reliability, or cost (Neves-Moreira et al., 2016). The concepts of logistics in operations research continue to gain significant benefits because of advancement and efficiency in modern science and technology. Manufacturing companies strive to optimise their product costs by lowering overhead costs. For example, in the energy and mining industries, transportation costs account for a substantial portion of the expenses.

According to Bienstock and Munoz (2015), freight accounts for one-third to two-thirds of the total cost of logistics, as a result, even a minor improvement in transportation efficiency would increase the company's net profit. Furthermore, transportation problems cannot be discussed without considering the economic link between demand and supply, which is critical to understanding the basic needs or problems that will be encountered.

1.2 Research Motivation

Transportation problems have many potentials as industries strive to reduce the cost of distribution of finished goods or essentials and move them to consumers at the lowest possible cost. However, there are some difficult issues with transportation that must be reviewed and addressed appropriately. In the traditional transportation problem, there is usually only one forwarder who moves goods from the manufacturer to the consumer while attempting to keep costs as low as possible. This research will look at several forwarders who transport goods from manufacturers to consumers to reduce transportation costs. As a result, the starting point of the research was previously to develop a systematic approach to improve on Vogel's Approximation Method (VAM), a well-known method for finding an initial basic feasible solution (IBFS) to a transportation problem that has the advantage of shortening the time required to compute the overall optimal solution, but the aim of this research is now focused on improving the state-of-the-art method of finding the IBFS for transportation problems.

This research initially addressed VAM's limitation of arbitrarily making allocations when there is a tie in maximum penalty and minimum cost. The previously proposed method Tiebreaker Vogel’s Approximation Method (TBVAM) (Madamedon et al., 2022), improves on the initial basic feasible solution of VAM by systematically breaking ties at several levels. However, the initial basic feasible solution of TBVAM was not always better than the ones from the state-of-the-art methods such as Total Opportunity Cost Matrix-Vogel’s Approximation Method (TOCM-VAM). Therefore, this research aims to improve on the state-of-the-art method TOCM-VAM by proposing the Total Opportunity Cost Matrix-Tiebreaker Vogel’s Approximation Method (TOCM-TBVAM) and an extension method of inducing ties at the maximum penalty cost known as Total Opportunity Cost Matrix-Tiebreaker Vogel’s Approximation Method Threshold (TOCM-TBVAM-TH). TOCM-TBVAM-TH applies a percentage threshold interval on the maximum penalty cost to induce ties and has the advantage of allowing several pathways in the solution search process and provides, on average, much more efficient IBFS.

1.3 Research Questions

The focus of this transportation problem research is to provide answers to the following questions:

1. Can an approach be proposed to address TOCM-VAM's limitations to improve the initial basic feasible solution?
2. What are the potential benefits of this proposed method that addresses TOCM-VAM’s limitations?
3. Would the possibility of creating ties using a percentage threshold improve the initial basic feasible solution of TOCM-TBVAM?

To address these questions, two batches of 10,000 transportation problems were simulated using the statistical package R, and 35 well-established benchmark transportation problems were taken from previously published literature on the field.

1.4 Aims and Objectives

Aim of the Research

As organisations continue to use Operations Research to manage their businesses, optimisation often comes into play, as they look to reduce overhead costs and maximise profits using their resources, including time. As a result, the goal of this research is to propose a new optimisation method that modifies and improves on TOCM-VAM in the decision-making process to enable it to solve transportation problems, more specifically, Demand and Supply Transportation Problems.

Note: in the context of this study, these transportation problems refer to both demand and supply transportation problems.

Research Objectives

The following are the main objectives of this research:

To present an overview of transportation problems in operations research in the form of a literature review.

To draw attention to the shortcomings of the most efficient method used to solve transportation problems known as TOCM-VAM.

To discuss and evaluate various published methods of finding an initial basic feasible solution.

Propose a novel approach that breaks ties in the allocation decision-making process and improves the performance of TOCM-VAM.

To put the proposed method into practice, using simulation problems, evaluate and test it comparing the results to the current state-of-the-art algorithm.

1.5 Expected Contributions to Knowledge

Although numerous ways have been utilised in the past to solve transportation problems with potential benefits, some of these methods include utilising the minimum cost method, using the maximum penalty cost, using the maximum mean, and using the cost cells' standard deviation. The key contributions to knowledge in this research are as follows:

- A method is proposed to improve TOCM-VAM's IBFS by breaking ties at several levels such as maximum penalty cost, minimum cost, and quantity supplied.
- TOCM-VAM can be further improved for real-value data sets by considering a user-defined percentage range to be taken as indicative of ties taking place using a novel proposed method, TOCM-TBVAM-TH.
- The results also indicate that TOCM-VAM can be improved on the number of wins on average by about 2% when we consider a range of percentage ranges with minimum effort which is not expensive to run.

1.6 Organisation of the thesis

Chapter 2

Provides a critical review of transportation problems in demand and supply in the context of optimisation. A wide-ranging overview of the current literature in obtaining an initial basic feasible solution for transportation problems, their advantages, and shortcomings, and identifying the gaps in the literature, as well as the reasons as to why the literature review is significant to the research.

Chapter 3

Presents the mathematical formulation of transportation problems, as well as the two types of transportation problems (balanced and unbalanced). Justification of why and how to find the initial basic feasible solution using the standard methods of finding IBFS, namely, Northwest Corner Method, Minimum Cost Method, and Vogel's Approximation Method. The optimal solutions to transportation problems and their benefit are discussed, including a review of using the Stepping-Stone approach and the Modified Distribution Method (MODI) to test for optimality.

Chapter 4

This chapter will present the starting proposed algorithm (TBVAM) and the link to the more recent proposed algorithm (TOCM-TBVAM), which is a method that improves on TBVAM and improves or has the same IBFS as TOCM-VAM. Two examples of transportation problems with integer costs are also presented. One example shows the procedures of TOCM-TBVAM without encountering any ties and thus performs in the same way as TOCM-VAM, while the other example breaks ties at several levels and the IBFS is compared to TOCM-VAM, which breaks ties arbitrarily.

Chapter 5

This chapter describes how the datasets for two batches of 10,000 simulated transportation problems were generated to be used as a real-valued cost to test the efficiency of the proposed method where banding transforms the continuity to induce ties in the algorithm, as well as the application of TOCM-TBVAM-TH and TOCM-VAM on the generated transportation problems.

Chapter 6

This chapter will discuss the findings and results of the comparison between TOCM-TBVAM-TH and TOCM-VAM. This chapter also addresses the computational complexity of the proposed algorithm against the state-of-the-art method TOCM-VAM and VAM, statistical analysis on the results obtained and the improvement on the state-of-the-art method as well as the overall effectiveness and applicability of the proposed methods are further analysed.

Chapter 7

This chapter will draw together all the other Chapters in this thesis and present a conclusion, the novel contribution to knowledge as well as the limitations and opportunities for future work.

CHAPTER TWO

Literature Review

2.1 Introduction

This chapter provides a critical review of transportation problems in demand and supply in the context of optimisation, an in-depth review of the current literature in obtaining an initial basic feasible solution for transportation problems, their advantages, and shortcomings, and identifying the gaps in the literature, as well as and the reasons as to why the literature is significant to the research.

2.2 Transportation Problem

It was around 1947 that George B. Dantzig first conceived linear programming problem while working as a mathematical advisor to the United States Airforce Comptroller on developing a mechanised planning tool for a time-stage deployment, training, and logistical supply program (Bazaraa et al., 2009). Although T.C. Koopmans coined the term "Linear Programming" in 1948, it was Dantzig who first published the "simplex method" for solving linear programs.

Linear programming initially referred to plans and schedules for training, logistical supply, and man deployment. However, linear programming is now used to plan all economic activities, such as the transportation of raw materials and products between factories, the sowing of various crop plants, and the cutting of paper rolls into shorter ones in sizes specified by customers (Matousek and Gärtner, 2006). Linear programming is concerned primarily with minimising or maximising (Optimization) a linear function while satisfying a linear equality and/or inequality constraint. According to Luenberger (1973), the popularity of linear programming stems primarily from the formulation phase of analysis rather than the solution phase—and for good reason, as their problem formation is simple. Linear programming is a technique for obtaining an optimal solution when we have limited resources and many competing candidates who want to consume the limited resources in a specific proportion (Murthy, 2005, Dantzig, 2016). Linear programming is one of the most versatile, powerful, and useful decision-making techniques. Sultan (2014) claimed that this and other applications aided in the victory of several major war battles.

Setting up an efficient and flexible logistics network, as well as defining its planning and operational processes is one of the most difficult challenges in the transportation industry (Neves-Moreira et al., 2016). The Second World War (1939–45) saw many changes, and it became clear that planning and coordination among many projects, as well as efficient utilisation of limited resources, were essential (Bazaraa et al., 2009). The distribution of man and equipment to destinations from various sources was one of the most significant challenges.

According to Ahmed et al. (2016a), transportation modelling is a technique used to plan the transportation of supplies from various sources to various destinations, therefore, the primary application of the transportation problem is to reduce the cost of distributing products or goods from various factories or sources to a variety of warehouses or destinations (Hitchcock, 1941).

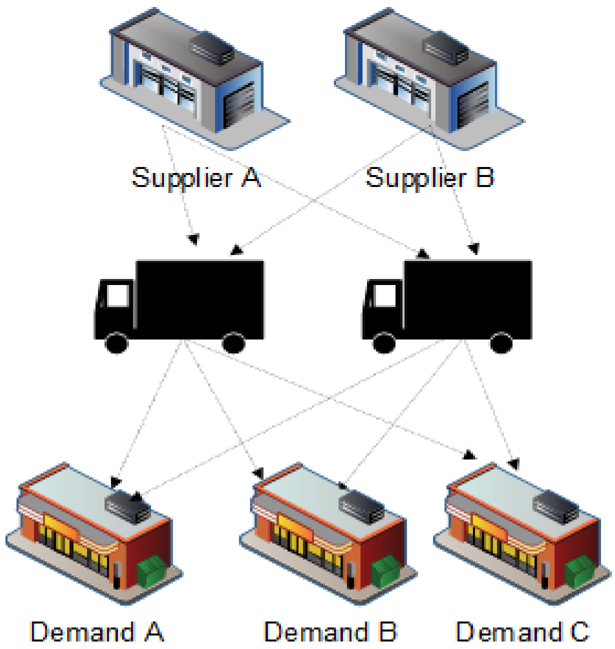

Figure 1: A classic Transportation Problem showing more than one forwarder from suppliers to demand points (Stein and Sudermann-Merx, 2018)

In the real world, as shown in Figure 1, each forwarder seeks to minimise transportation costs while sharing supply and demand constraints with other forwarders. It is expected that the precise values of transportation costs are known in transportation problems; however, in the real world, the exact price may not be certain due to uncontrollable factors (Uddin and Huynh, 2016). Hitchcock (1941) explained that the cost of a product in a specific city will sometimes vary depending on which factory supplies it due to freight rates and other factors. In this study, we look at transportation costs that are known and precise.

2.3 Real-life Cases

A real-life example was Procter and Gamble's distribution in the 1990s when they saved a significant amount of money by restructuring the supply chain in their North

American centre. Procter & Gamble deregulated the trucking industry, resulting in lower transportation costs and lower prices for goods. Secondly, the products were more compacted, allowing more goods to be transported per truckload. Another way this aided product quality was that they were more dependable. Similarly, the product Lifecycle was reduced, resulting in more frequent plant changes, and finally, product cost was reduced due to excess capacity. All of this resulted in Procter & Gamble saving approximately $200 million per year (Camm et al., 1997).

Brenton Barr provided another example of a transportation problem, citing transportation costs as a factor in the location of the Soviet Wood Processing Industry. Due to Roundwood surpluses and deficits, the study's goal was to allocate Roundwood shipment from supply sources to deficit areas to reduce total transportation costs. Barr used the transportation problem to determine the best pattern of flow between supply and demand regions (Skorin-Kapov et al., 1996).

2.4 Current Literature on finding IBFS to Transportation Problem

It is essential to obtain an IBFS for transportation problems, particularly large transportation problems, to find the optimal solution. There are several heuristic approaches for obtaining an IBFS; however, while some heuristics can obtain an IBFS efficiently, the solution they find is sometimes not optimal in terms of the total cost. Other heuristics, on the other hand, may not produce an IBFS as quickly, but the solution they find is often quite good in terms of minimising the total cost (Anderson et al., 2018). As a result, many studies have implemented new approaches for determining the IBFS, including several VAM modifications. In these varied approaches, it is worth noting that some have considered a tie in the maximum penalty, while others did not. Additionally, some have considered minimum cost tie and, supply and demand tie, while others have not. Consequently, this literature review can be divided into groups in which researchers considered ties in their approaches while others assigned arbitrary allocations when there are ties.

The Harmonic Mean Method (HMM), for example, does not consider ties. Palanivel and Suganya (2018) apply the harmonic mean for each row and column, identifying the maximum cost among each row and column and then makes allocations to the minimum cost cell, and in case of a tie in minimum cost, it makes arbitrary allocations to the minimum cost.

Although this approach claims to produce IBFS solutions as VAM and sometimes the optimal solution, which are some of its benefits, it does have certain drawbacks. For example, the solution to the transportation problems was only compared to the three common methods of finding the IBFS, i.e., Least Cost Method, Northwest Corner Method, and VAM, which will be discussed in detail in the next chapter, and the dimension of the transportation problem was quite small in contrast to real-world cases. In addition, the lack of considering tie at minimum cost is another drawback as it would take several paths to see an improvement to the IBFS. In their approach known as Improved Vogel's Approximation Method (IVAM), Korukoğlu and Ballı (2011) did not consider a tie. This approach entails the three highest penalty costs as well as the total opportunity cost. This approach was evaluated on a few large size dimension transportation problems, and it was discovered that VAM was only better on small dimension transportation problems. When compared to VAM, this method obtains significantly more efficient IBFS for large-size transportation problems. Nevertheless, because of the calculation required to determine the total opportunity cost and the highest three penalty costs, as well as the additional two alternative allocation costs, this approach has a disadvantage in terms of higher computational complexity. Additionally, the large transportation problem used in this approach did not include real-value cost.

Ahmed et al. (2016b) utilise an allocation table to select the minimum odd cost from all cells and divide any even cost cells until they become an odd cost, but ties in the minimum odd cost cells are resolved by looking at the supply/demand point with the lowest quantity. Although this method is effective for obtaining the IBFS, its shortcoming is that the cost of transportation from each source to the destination cannot be reduced to a decimal number since it is impossible to predict whether the decimal number will be odd or even during the iteration. This indicates that implementing real-value cost is an area that needs additional research.

Islam et al. (2012) extended this approach to the Total Opportunity Cost Table (TOCT). To calculate the penalty cost in TOCT, we subtract the minimum cost from the maximum cost. However, it breaks ties arbitrarily in the maximum penalty and minimum cost. Although this approach was limited to a single example, it has the same IBFS as VAM, which was the same as the optimal solution. The disadvantage of this approach is that it does not consider ties at some levels, and no work was done on large dimension transportation problems or the usage of real-valued cost. Similarly, Hosseini (2017) proposes three new algorithms known as Total Difference Method (TDM1), TDM2, and Total Difference Least Square Method (TDSM), all of which arbitrarily break ties. TDMI only uses the differences in the minimal cost between the rows and does not compute the columns. This is calculated by removing each minimum cost from each row, then adding the sum of all the differences and allocating it to the row with the maximum cost sum. This reduces computational time to attain an IBFS because it just uses rows; however, the drawback of TDM1's IBFS is that it is not as efficient as VAM for small-scale transportation problems. Furthermore, when TDM1 was computed using a large dimension transportation problem, the results indicate that TDM1 is more efficient on average since it provides a better IBFS. TDM2 computes the total difference by combining rows and columns, but neither TDM2 nor TDSM was as efficient as VAM with either a small or large dimension transportation problem.

Similarly, in an attempt to exploit the opportunity cost by better taking into account the variation within the entries in the cost matrix, Akpan et al. (2015) modified VAM by using the idea of obtaining the standard deviation of the rows and columns and make allocations to the minimum cost cell with the greatest costs-standard deviation without taking into consideration when ties occur in the standard deviation of the row and column respectively. This approach's disadvantage is that it is not any better than VAM. Yet, it has the advantage of providing an IBFS that can be utilised to find the optimal solution with less computational time. Kirca and Şatir (1990) worked on transportation problems by generating the Total Opportunity Cost Matrix (TOCM) which is obtained by subtracting the minimum cost from each row and column and then adding the result to make a new transportation problem matrix. Mathirajan and Meenakshi (2004) extended TOCM with VAM to achieve a more efficient IBFS. This approach known as Total Opportunity Cost Matrix-Vogel's Approximation Method TOCM-VAM is computed by applying VAM on TOCM. Using the transportation problem generated by TOCM, the penalty is calculated by subtracting the minimum cost from the next minimum cost. Should there be two or more minimum costs, then the penalty is equal to zero. Then maximum allocation according to the supply and demand constraints is made to the maximum penalty cost cell with the minimum cost.

In case there is a tie in the maximum penalty, allocation is done arbitrarily, and in case there is a tie in minimum cost, again allocation is done arbitrarily.

These arbitrary allocations in maximum penalty and minimum cost are the shortcomings of this approach. Although this method produces more efficient IBFS than other existing methods which is an advantage as it requires less computational time to get the optimal solution, it would be good to see how it performs with large-size dimension transportation problems and real-valued cost.

Total Opportunity Cost Matrix Sum (TOCM-SUM) is another method of finding an IBFS of transportation problems (Khan et al., 2015). To achieve this, after obtaining the TOCM, pointer costs are assigned which are the sum of the cost in the rows or the cost in the column. Maximum allocation according to the demand and supply constraints is then given to the minimum cost cell with the highest pointer score. In this method, in the case of ties, allocations are done arbitrarily. This method has the advantage that it is easy to compute and understand, the shortcoming outweighs the advantages as this method makes arbitrary allocation when there is a tie. In addition, the IBFS obtained by this method was on a small size dimension, and in most cases, the result obtained was the same as TOCM-VAM. To make a variety to TOCM-SUM, another method of finding the IBFS known as the Total Opportunity Cost Matrix-Median Extreme Difference Method (TOCM-MEDM) was proposed (Hossain et al., 2020). This TOCM-MEDM was a modification on MEDM but first obtains TOCM. Whereas other methods subtract the two minimum costs to achieve the penalty, TOCM-MEDM achieves the penalty by subtracting the minimum cost from the maximum cost, and then makes allocations arbitrarily in cases of ties in the maximum penalty, however in cases of ties in minimum cost, it allocates to the cost cell where maximum allocation can be obtained. The advantage of this method is that it is easy to compute to get the IBFS of transportation problem, however, the arbitrary allocation when there is a tie in maximum penalty is a shortcoming with this method. Similarly, although the IBFS obtained were the same as other existing methods, the transportation problem was also on small size dimension. Total Opportunity Cost Matrix-Highest Cost Difference Method (TOCM-HCDM) utilises the TOCM approach and arbitrarily breaks ties, this approach calculates the penalty by subtracting the next maximum cost from the maximum cost, which prevents the penalty from being assigned a zero cost. This method has the advantage of being simple to understand and compute, but the IBFS it obtained were not efficient or comparable to other methods.

In contrast to the preceding approaches (which did not consider ties), some alternative methods consider ties when determining the IBFS of a transportation problem. Approaches such as Total Opportunity Cost Matrix-Minimal Total (TOCM-MT) by (Amaliah et al., 2019) use the TOCM, which is the result of subtracting the minimal cost from each row and column and then adding them to form a transportation problem matrix, and break ties by assigning allocation to the minimum cost cell when there is a tie in the maximum penalty, then to the maximum overall cost when there is a tie in the minimum cost, and finally to the greatest quantity demanded when there is a tie in maximum overall cost. The downside to this approach is that it requires more computational time and is only applicable to small-size dimension transportation problems. It does, however, have the advantage of obtaining a more efficient IBFS because it breaks ties at specific levels (maximum penalty, minimum cost, and maximum overall cost). Similarly, the Abdul Sum Method (ASM) method by Murugesan and Esakkiammal (2019) was proposed to work on getting the IBFS of transportation problems. ASM applies the row minimum subtraction (RMS and column minimum subtraction (CMS) which then gives a reduced cost matrix (RCM). Each RCM then must have a zero-entry cell, this is the cell with a zero cost. The main concept is to make maximum allocations to a zero-cost cell, if a tie occurs in the zero-cost cell, allocation to the cost cell with the maximum sum of the cost cell in the row or column. If a tie occurs, then any zero-cost cell is arbitrarily chosen for allocation. This is because the ASM method finds it difficult to identify the appropriate zero-entry cell for allocation in an RCM in case of a tie among certain zero-entry cells. This tends to be a drawback of this approach as it did not break ties at all levels. Therefore, to improve on the shortcomings of ASM, the Improved Abdul Sum Method (IASM) was proposed to address the tie-breaking method of ASM. Here IASM counts the total number of zeros cell (excluding the selected one in its row and column and then make maximum allocations to zero-entry cell for which the number of zeros counted is minimum. IASM may then produce a better IBFS than ASM. The drawback of this approach is that the improvements are mostly on small dimension transportation problems and unbalanced transportation problems, where the total quantity demanded is different from the total quantity supplied. This tie-breaking method shows the benefit of why ties need to be broken in a specific way. Juman and Hoque (2015) proposed an approach for computing penalty cost using only the columns and ignoring the rows. JHM breaks ties in the minimum cost by assigning the allocation to the cell with the highest quantity demanded. Another method, known as the New Method, finds the penalty cost and excludes it by using the row difference (Juman and Nawarathne, 2019). The benefit of these two approaches is that they sometimes create IBFS identical to VAM, the disadvantage is that they only work on small-size dimension transportation problems. The work from Amaliah et al. (2020) modified

JHM with a method known as Bilqis Chastine Erma (BCE) in ties by adding a procedure to move the excess quantity demanded to the second minimum cost cell. BCE worked on small size dimension transportation problems and not all the IBFS was better than other existing researched methods. For example, some of the IBFS were higher than methods such as VAM, TOCM-MT, and TDM1. However, BCE has the advantage of getting the optimal solution faster in some cases but opens further research to be undertaken for transportation problems with large size dimension that has large quantities of supply and demand.

Some researchers have explored methods to improve VAM and break ties. One example is the Advanced Vogel's Approximation Method (AVAM), which uses the next two minimum costs to avoid assigning allocations to a greater cost in the next iteration (Das et al., 2014). Although AVAM is not as efficient as VAM, it can also provide an IBFS that is sometimes lower than VAM, but on average, VAM produces a better IBFS. Additionally, it would have been better to see the performance of the IBFS on AVAM on large-size dimension transportation problems and compare it to other existing methods. A recently improved edition of this algorithm also identifies the two minimum costs in each row and column as in VAM. It does, however, deal with a tie in the penalty cost by allocating it to the minimum cost cells. Similarly, if there is a tie in the minimum cost cells, allocation is made to the demand point where the maximum quantity supplied may be satisfied. In case of a tie in the demand points, allocation is made from the cell having the most available supply quantity. Khan et al. (2015b) proposed a method that works on TOCM, but instead of allocating the maximum penalty, it makes an allocation to the minimum cost with the minimum penalty, and if a tie occurs, allocation is made arbitrarily. This method has the advantage that it is easy to understand and compute, however, the small size of the dimension of the transportation problem is a shortcoming. Furthermore, the IBFS obtained were the same as other known methods. Islam, Khan, Uddin, and Malek (2012) proposed a method that works on TOCM. This approach, known as the Extreme Difference Method (TOCM-EDM), allocates the minimum cost with the maximum difference after finding the penalty by subtracting the two maximum costs from the row and column, respectively. If the maximum penalty is a tie, the penalty will be zero. If there is a tie in minimum cost, allocation is done arbitrarily. TOCM-EDM has the advantage of being simple to understand and compute, but it has the disadvantage of producing the same IBFS as other existing approaches and being tested on a small-size dimension transportation problem. Lekan et al. (2021) proposed a method known as the Maximum Difference Extreme Difference Method (MDEDM) to find the IBFS of transportation problems. This method uses the maximum difference of the rows which is obtained by subtracting the immediate maximum cost from the maximum cost and then calculates the extreme difference of the columns by subtracting the minimum cost of the column from the maximum cost of the column and then makes an allocation to the minimum cost with the maximum difference. In the case of ties, allocations are made to the minimum cost cell at the topmost row and the extreme left corner. The advantage of this method is that it is to implement and does deal with ties at a level, but the drawback is that the IBFS obtained are the same or higher than other existing methods of finding IBFS. In addition, the dimension of the transportation problem solved was of a small-size dimension.

Abdelati (2023) proposed an algorithm known as the Cost-Quality Method (CQM). This algorithm identifies the lowest cost in the cost matrix and divides all cost cells by it. It then determines the maximum amount that can be assigned to each row and column and divides each demand cell by the maximum demand values to generate another matrix allocation table. These two allocation matrix tables are then multiplied to form a new allocation table. The allocations are then made to the cost cell with the highest values. We update the allocation tables with the original cost matrix values to calculate the total transportation cost. This method has the advantage of producing an initial basic feasible solution that is lower than the LCM and VAM, however, the shortcomings are that it is not as efficient as other existing methods including the state-of-the-art method TOCM-VAM, and it was only used for small-size dimension transportation problem.

Another algorithm known as the Demand-Based Allocation Method (DBAM) was proposed by Ackora-Prah et al. (2023). This algorithm uses the minimum demand with the lowest cost values to make allocations. Although this method has the disadvantage of lacking an expected way of breaking ties, which it does arbitrarily during the decision-making process, it does have the advantage of generating some efficient results when compared to other existing methods.

Furthermore, the use of the Mean Absolute Deviation Method was proposed by Thanoon (2022). This algorithm computes the mean absolute deviation of the rows and columns and allocates to the cell with the lowest cost and the highest mean absolute deviation. This method has the advantage of producing an efficient IBFS that is the same as VAM, but it has only been tested on one small size dimension transportation problem, and most importantly, it does not mention how to break ties in the decision-making process, which is a shortcoming of this method.

To compare the various and more recent methods of obtaining IBFS to transportation problems, after rigorous experimental computation, Sultana et al. (2022) maintained that on average TOCM-VAM was more efficient as it produced better IBFS than all other existing methods. It also maintained that for small-size dimension transportation problems, FSTP and the Modified Distribution Method produced the same optimal solution as the Modified Distribution Method. Therefore, these findings motivate the researcher to find a better method that would improve on the state-of-the-art TOCM-VAM of finding an IBFS that is either as efficient as TOCM-VAM or better. Also, from the literature review, since only a few levels of ties are common which tends to be a drawback even in the most efficient method, the main focus of this research is to explore how the TOCM-VAM would perform in a large-size dimension transportation problems that use real cost values as against TOCM-TBVAM-TH, that creates ties in the maximum penalty since it would be extremely difficult to have ties in real-valued cost transportation problems.

2.5 Overview and comparison of five state-of-the-art Benchmark Algorithms

Five different algorithms for finding an initial basic feasible solution are compared with TOCM-VAM, TOCM-TBVAM and the extension of TOCM-TBVAM known as TOCM-TBVAM-TH. The majority of these algorithms have been evaluated on small-scale transportation problems that can be solved by hand. The largest dimension size of transportation problems on which these algorithms were tested was a 7 x 8 matrix, while the smallest was a 3 x 4 matrix. As a result, they are typically basic transportation problems with few computations but several iterations, such as VAM, TOCM-SUM, and TOCM-VAM. Furthermore, when compared to other algorithms, these benchmark algorithms have consistently ranked as good or the best. For example, when VAM is compared to the other two algorithms like LCM and NWCM, VAM tends to be the best. Likewise, other algorithms, such as TOCM-MEDM, MDEDM, and JHM, have produced efficient initial basic feasible solutions. However, they all have drawbacks in that they were only tested on small-size transportation problems with integer costs. **Table 1** shows a summary of the main features, advantages, and shortcomings of some of the state-of-the-art algorithms for solving transportation problems.

Table 1. Summary of the main features, advantages, and shortcomings of some state-of-the-art algorithms that solve transportation problems.

TP Algorithms	Main features	Advantages	Shortcomings
VAM (1958)	Uses two minimum costs to compute penalties. Makes allocations to the minimum cost cell with the maximum penalty.	Produces on average a better IBFS than some standard methods, e.g., LCM and NWCM.	Takes more time than LCM and NWCM to achieve the IBFS as it involves lots of iterations process. Makes arbitrary allocations in case of ties.
TOCM-SUM (2015)	Uses TOCM to compute penalties. Appoints pointer scores to the sum of rows and columns. Makes allocations to the minimum cost cell with the highest pointer score.	Easy to compute and understand. Some of the IBFS obtained on average are the same or better than VAM's algorithm.	Makes arbitrary allocations in case of ties. Only tested on small-size dimension TP.
TOCM-MEDM (2020)	Uses TOCM to compute penalties. It subtracts the minimum cost from the maximum cost to obtain the penalty cost.	Easy to compute and understand. The IBFS obtained on average are the same or better than other standard methods such as LCM and NWCM.	Only considers 1 level of tie-breaking which is in the maximum penalty. Tested on only small size dimension TP.

JHM (2015)	It computes penalty cost with only the columns, ignoring the rows. Makes allocations to the minimum cost cell. It breaks ties in minimum cost by making allocations to cells with the highest quantity demanded.	Easy to compute and understand. Breaks tie at 1 level, i.e., ties at the minimum cost. IBFS obtained on average are the same or better than VAM.	Only works well on small size dimension TP. Only considers ties at 1 level in the decision-making process.
MDEDM (2021)	Uses maximum difference of rows by subtracting the two maximum rows cost. Uses the minimum difference of the column by subtracting the minimum cost from the maximum cost in the column.	Easy to compute and understand. Consider ties at 1 level. i.e., at maximum penalty only.	The IBFS obtained on average are not better than other existing methods. Works well on only small size dimension TP. Considers ties only at 1 level.
TOCM-VAM (2004)	Uses TOCM to compute penalties. Then apply VAM to TOCM. Makes allocations to maximum penalty with minimum cost.	Easy to compute and understand. IBFS obtained on average the same or better than all existing methods.	Does not consider ties in the decision-making process. Needs to be rerun to get a lower IBFS as it makes arbitrary allocations when there are ties. Not tested on real-valued cost.
TOCM-TBVAM (2023)	Uses TOCM to compute penalties.	Easy to compute and understand.	Takes more time to achieve IBFS as it considers ties at several levels.

TOCM-TBVAM (2023) (continued…)	Makes allocations to minimum cost in maximum penalty cell. Makes allocations to the minimum cost with the maximum mean cost. Makes allocations to the maximum quantity that can be supplied which has the minimum cost that has the maximum mean cost cell.	Considers ties at 3 levels. i.e., maximum penalty, minimum cost, maximum mean cost. Systematically break ties at several levels. Each level of tie-breaking, on average improves the IBFS compared to TOCM-VAM. The IBFS obtained are the same as TOCM-VAM when there are no ties, or better, on average, when there are ties.	
TOCM-TBVAM-TH (2023)	In addition to all steps in TOCMTBVAM, TOCM-TBVAM-TH uses multiple percentage thresholds to induce ties at the maximum penalty.	In addition to all the advantages of TOCM-TBVAM, the percentage threshold gives this algorithm opportunities to obtain a lower IBFS as it creates several pathways in the decision-making process. Tested on large-size dimension TP with real-valued cost.	It takes more time than other existing methods as it induces ties at the maximum penalty and breaks ties at several levels.

2.6 Summary

This chapter provided a thorough review of current research on finding the IBFS for transportation problems. It is maintained that TOCM-VAM is the most efficient approach, and it is also highlighted that arbitrary allocations often result in a less efficient IBFS due to ties at multiple levels in the iteration. Using real-valued cost on large size dimension transportation problem is also missing in most research; this, among other reasons, motivates the researcher to work on this area of transportation problem. The next chapter will present the mathematical formulation of transportation problems, as well as the two types of transportation problems (balanced and unbalanced), and the significance of finding the initial basic feasible solution and optimal solutions to transportation problems, including a review of using the Stepping-Stone approach and the Modified Distribution Method (MODI) to test for optimality.

CHAPTER THREE

Standard Methods of Finding the Initial Basic Feasible Solution

3.1 Introduction

The preceding chapter reviewed the literature and found that most studies on obtaining IBFS for transportation problems had various advantages, weaknesses, and limitations. It also highlighted the motivation for this research, which was to improve on the limitations of the state-of-the-art method (TOCM-VAM). This chapter covers the conventional approach for constructing a mathematical formulation procedure to represent transportation problems with their constraints. Furthermore, this chapter will discuss the two types of transportation problems, what the initial basic feasible solution and optimal solutions are, the three most used initial basic feasible solution, their significance, and a review of how to use the Stepping-Stone approach and the Modified Distribution Method (MODI) to test for optimality.

3.2 Mathematical Formulation for Transportation Problems

For a given transportation problem, the unit cost and constraints are specified by the following information:

- A set of m $supply\ points$ from which the goods are shipped. This is known as the supply point i which can supply at most s_i units.
- A set of n $demand\ points$ to which the goods are shipped. This is known as the demand point j which must receive at least d_j units of goods.
- Each unit produced at supply point i and shipped to demand point j incurs a variable cost of C_{ij}

Let

X_{ij} = Number of units shipped from supply point i to demand point j

Thus, the general formulation of a transportation problem is given as

$$Minimize\ Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij} \quad (1)$$

$$\sum_{j=1}^{n} x_{ij} \le s_i (i = 1,2, \dots., m)\ Supply\ Constraints \quad (2)$$

$$\sum_{i=1}^{m} x_{ij} \ge d_j\ (j = 1, 2, \dots, n)\ Demand\ Constriants \quad (2)$$

$$\mathrm{x_{ij}} \ge 0\ (\mathrm{i} = 1, 2, \dots, \mathrm{m};\ \mathrm{j} = 1,2, \dots, \mathrm{n}) \quad (3)$$

3.3 Balanced Transportation Problem

When the total amount of goods at the origin equals the total requirement at the destination, the transportation problem is said to be balanced. This is required to resolve the transportation problem (Harrath and Kaabi, 2018). If the total goods available are less than the requirements, an imaginary dummy availability with zero associated transportation cost is added (Girmay and Sharma, 2013, Vasko and Storozhyshina, 2011).

For a balanced transportation problem, the formulation is:

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j \quad (4)$$

3.4 Unbalanced Transportation Problem

An unbalanced transportation problem occurs when the total availability of goods at the origin does not equal the total requirements at the destination. This could be because there is more demand than supply, or vice versa (Vasko and Storozhyshina, 2011). If there is more availability at the origin than demand at the destination, an additional column is added to indicate the surplus supply with no transportation cost to obtain an initial basic feasible solution. If the total demand exceeds the supply, an additional row is added to represent unsatisfied demand (Vasko and Storozhyshina, 2011).

The mathematical formulation for unbalanced transportation problems is shown in equation 6.

$$\sum_{i=1}^{m} s_i \neq \sum_{j=1}^{n} d_j \tag{5}$$

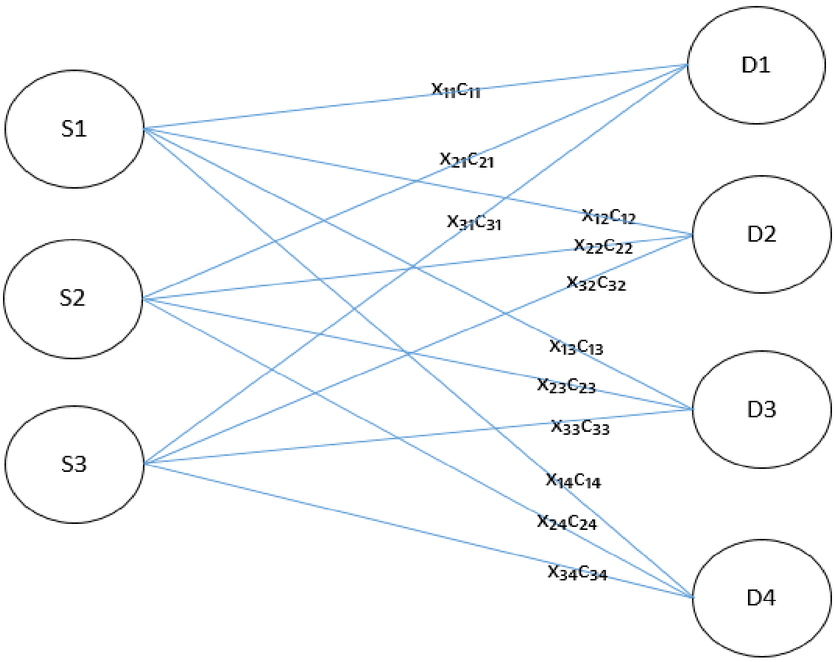

Figure 2: Graphical representation of transportation problems network flow from supply to demand.

As shown in Figure 2, the total number of goods to be transported from S1 to all destinations, i.e., D1, D2, D3, and D4 must be equal to $a_{1,}\ where\ S1 = a_{1,}\ and\ S2, S3 = a_{2\,,}\ a_3\ respectively$

Thus

$$x_{11} + x_{12} + x_{13} + x_{14} = a_1 \tag{6}$$

$$x_{21} + x_{22} + x_{23} + x_{24} = a_2 \tag{7}$$

$$x_{31} + x_{32} + x_{33} + x_{34} = a_3 \tag{8}$$

Also, the total goods delivered to D1, from all units must equal $b_{1,}\ where\ D1 = b_{1,}\ and\ D2,\ D3\ and\ D4 = b_{2,}\ b_{3,}\ b_{4,}\ respectively.$

Thus

$$x_{11} + x_{21} + x_{31} = b_1 \tag{9}$$

$$x_{12} + x_{22} + x_{32} = b_2 \tag{10}$$

$$x_{13} + x_{23} + x_{33} = b_3 \tag{11}$$

$$x_{14} + x_{24} + x_{34} = b_4 \tag{12}$$

3.5 Initial Basic Feasible Solution

An initial basic feasible solution is a significant step toward determining the optimal solution in general because it helps to reduce the number of iterations required to achieve the optimal solution (Amaliah, Fatichah and Suryani, 2020). As a result, as a first step toward the optimal solution, it is essential to find an initial basic feasible solution. As will be shown later, this proposed algorithm (TOCM-TBVAM) reflects achieving a lower cost as a foundation for finding the optimal solution by systematically breaking ties at several levels instead of arbitrary allocations in case of ties at multiple levels when compared to TOCM-VAM. Fundamentally, it is expected that the initial basic feasible solution must have (m + n – 1) basic independent parameters out of (m x n) parameters, where m is the origin and n is the destination. However, in cases where the number of allocations in an initial basic feasible solution is less than (m + n -1), it is referred to as a degenerate basic feasible solution; otherwise, it is referred to as a non-degenerate basic feasible solution (Murugesan and Esakkiammal, 2020; Muthuperumal, Titus and Venkatachalapathy, 2020).

3.6 Standard methods of finding IBFS to Transportation Problems

Typically, well-known methods such as the Northwest Corner Method (NWCM), Least Cost Method (LCM), or Vogel's Approximation Method (VAM) are used to obtain the initial basic feasible solution of any transportation problem, and then MODI also known as Simplex Method is used to check the optimality of the given transportation problem.

3.6.1 Northwest Corner Method (NWCM)

The NWCM is a method to calculate a transportation problem's initial basic feasible solution. It starts in the northwest (top left) corner of the cost cell and assigns as many units as possible of goods to this cost cell according to the demand and supply constraints. After that, the quantities in the supply and demand columns are changed accordingly. Once the supply for the first row is depleted, a transfer down to the second row in the second column is necessary. However, if the demand for in first row cell is met, the next cell in the second column must be moved horizontally. This process will continue until all the available goods have been allocated. Furthermore, because the

NWCM does not account for the minimum cost of shipping when making the allocation, the IBFS obtained is frequently extremely high (Winston & Goldberg, 2004).

3.6.2 Least Cost Method (LCM)

The LCM allocates the maximum possible quantity between supply and demand on the row and column using the variable with the lowest shipping cost. Cross out either the exhausted row or column and reduce the supply or demand of the crossed-out rows or columns once the allocations have been satisfied. Then choose a cell with the lowest cost from the cells that are not in a crossed-out row or column and repeat the process until all supply and demand are met. If a row fulfils both supply and demand constraints, just one row or column should be crossed out, not both. Furthermore, because LCM considers transportation costs when making allocations, it often yields a lower IBFS than NWCM (Lekan, Kavi and Neudauer, 2021).

3.6.3 Vogel's Approximation Method (VAM)

The VAM is an iterative approach that considers unit penalty and can produce a better IBFS that is sometimes the same as the optimal solution. However, providing a solution for a large size dimension transportation problem can take a long time (Winston & Goldberg, 2004). VAM starts by calculating a 'penalty' for each row or column, which is equal to the difference between the two lowest costs in the row and the two lowest costs in the column. Then choose the row or column with the highest penalty and select the cost cell in the row or column with the lowest cost to make allocations. Cross off the row or column when supply or demand has been fulfilled, just as performed with the NWCM and LCM, and the process repeats until all supply and demand points are exhausted.

3.7 Computational Time Complexity

We consider demand and supply transportation problems with s suppliers and d demand points. The problem is written as a $(s \times d)$ cost matrix as shown in **Table 2**.

Table 2. Generic demand and supply transportation problem.

In **Table 2,** c_{ij} is the cost per unit of sourcing a generic product from supplier i to demand j. d_j (demand of j) is the total number of units of the product required by demand point j. s_i (capacity of i) is the total number of units of the product that supplier i is able to supply.

	demand 1	demand 2	demand 3	supply
supplier 1	c_{11}	c_{12}	c_{13}	s_1
supplier 2	c_{21}	c_{22}	c_{23}	s_2
supplier 3	c_{31}	c_{32}	c_{33}	s_3
supplier 4	c_{41}	c_{42}	c_{43}	s_4
demand	d_1	d_2	d_3	

Henceforth, we will use n to denote the size of a transportation problem where $n = s + d$. In addition, for simplicity of the calculations and without loss of generality, unless stated otherwise, for all transportation problems considered, the number of supply points is equal to the number of demand points, i.e. $s = d$ (a square table). Because $s = d$ and $n = s + d$, $n \geq 2$ and n is even.

3.7.1 Time Complexity of NWCM

NWCM is one of the simplest algorithms to find an IBFS for a transportation problem. The method is computationally efficient but does not consider the costs of transportation even though this is what it seeks to minimize.

Unlike VAM and its derivatives, NWCM does not compute any penalties or look into costs to decide on assignments. Thus, the total number of decisions the algorithm executes is equal to the number of assignments, i.e. $n - 1$ (recall that, with no loss of generality, we are considering $n = s + d$).

As a result, the time complexity of NWCM is linear and equal to the number of assignments it performs $O(n - 1) = O(n)$.

3.7.2 Time Complexity of LCM

Like the NWCM, the LCM does not compute any penalties to decide on assignments. However, at each assignment, it does search for the minimum cost into the cost matrix to choose the assignment cell. Recall that we are assuming $n = s + d$ and $s = d$.

3.7.2a Worst-case scenario time complexity $O(f(n))$ for LCM

Considering the first assignment for LCM:

Time complexity to search for the minimum cost at each cell of the cost matrix is equal to the number of cells in the matrix which is equal to $s \times d = \frac{n}{2} \times \frac{n}{2} = \left(\frac{n}{2}\right)^2$.

Assuming a worst-case scenario, the number of rows and columns will only decrease by 1 at a time and will alternate, for example $\{(s, d), (s - 1, d), (s - 1, d - 1), (s - 2, d - 1), \dots, (1, 1)\}$.

Hence, the time to complete all $n-1$ assignments is:

$$\sum_{k=1}^{n}\left(\frac{k}{2}\right)^2+\sum_{k=2}^{n}\left[\left(\frac{k}{2}-1\right)\left(\frac{k}{2}\right)\right]=\sum_{k=1}^{n}\left(\frac{k}{2}\right)^2+\sum_{k=2}^{n}\left[\left(\frac{k}{2}\right)^2-\left(\frac{k}{2}\right)\right]=$$

$$=\sum_{k=1}^{n}\left(\frac{k}{2}\right)^2+\sum_{k=2}^{n}\left(\frac{k}{2}\right)^2-\sum_{k=2}^{n}\left(\frac{k}{2}\right)$$

$$=\sum_{k=1}^{n}\left(\frac{k}{2}\right)^2+\sum_{k=1}^{n}\left(\frac{k}{2}\right)^2-\left(\frac{1}{4}\right)-\left(\sum_{k=1}^{n}\left(\frac{k}{2}\right)-\left(\frac{1}{2}\right)\right)=$$

$$=2\sum_{k=1}^{n}\left(\frac{k}{2}\right)^2-\left(\frac{1}{4}\right)-\left(\sum_{k=1}^{n}\left(\frac{k}{2}\right)-\left(\frac{1}{2}\right)\right)=$$

$$=\frac{1}{2}\sum_{k=1}^{n}k^2-\left(\frac{1}{4}\right)-\left(\frac{1}{2}\sum_{k=1}^{n}k-\left(\frac{1}{2}\right)\right)=$$

$$=\left(\frac{n\,(n+1)(2n+1)}{12}\right)-\left(\frac{1}{4}\right)-\left(\frac{n\,(n+1)}{4}\right)+\left(\frac{1}{2}\right)\Leftrightarrow \boldsymbol{O(n^3)}$$

The slope of the worst-case scenario time complexity for LCM ($\log(n)$ *vs* $\log(\text{time complexity})$) is $\frac{\log(n^3)}{\log(n)}=3$.

3.7.2b Average time complexity $\theta(f(n))$ for LCM

Note that $O(n^3)$ is an upper bound for the time complexity of LCM. In practice, a single assignment can simultaneously eliminate both a demand point and its supplier from the cost matrix. This will happen every time that the remaining capacity of supply is equal to the demand required at the point of assignment and it is expected to happen multiple times during the execution of LCM. Consequently, on average, some assignments may reduce the size of the cost matrix by 2 (one row and one column simultaneously) unlike the aforementioned worst-case scenario. For instance, for a giving assignment from supply i to demand j, if $s_i = d_i$, both the respective supply i (row) to demand j (column) will be removed from the cost matrix and the problem size reduces from say n to $n-2$ in one assignment.

For the worst-case scenario, the time complexity to search for the minimum cost at each cell of the cost matrix is equal to $s \times d = \frac{n}{2} \times \frac{n}{2} = \left(\frac{n}{2}\right)^2$. In practice, because of the above-mentioned randomly occurring simultaneous reductions of rows and columns from the cost matrix, it is reasonable to assume (approximate) that the average time complexity to search for the minimum cost at each cell of the cost matrix is sublinear and approximately $\log\left(\left(\frac{n}{2}\right)^2\right)$.

Considering the first assignment for LCM, the time to complete all $n-1$ assignments is:

$$\sum_{k=1}^{n-1} \log\left(\left(\frac{k}{2}\right)^2\right) = \sum_{k=1}^{n} \log\left(\left(\frac{k}{2}\right)^2\right) - \log(\frac{1}{4})$$

The summation term can be written as:

$$\sum_{k=1}^{n} \log\left(\left(\frac{k}{2}\right)^2\right) = \sum_{k=1}^{n} \log\left(\frac{k^2}{4}\right) = \sum_{k=1}^{n} \log(k^2) - n\log(4)$$

and

$$\sum_{k=1}^{n} \log(k^2) = 2\sum_{k=1}^{n} \log(k) = 2\ \log\left(\prod_{k=1}^{n} k\right) = 2\log(n!)$$

Using Stirling's approximation (or Stirling's formula) (Romik, 2002)

$$\log(n!) = n\log(n) - n + O(\log(n))$$

Therefore,

$$\sum_{k=1}^{n} \log(k^2) - n\log(4) = n\log(n) - n + O(\log(n)) - n\log(4) \Leftrightarrow \boldsymbol{O(n\log(n))}$$

For LCM, the slope of $\log(n)$ *vs* log(time complexity) is:

$$\frac{\log(n\log(n))}{\log(n)} = \frac{\log(n) + \log(\log(n))}{\log(n)} = 1 + \frac{\log(\log(\mathrm{n}))}{\log(n)}$$

3.7.3 Time Complexity of VAM

The fastest algorithm to find the two largest numbers in a vector with k elements has a time complexity equal to $O(k)$ as shown in pseudocode 1 below.

Pseudocode 1: Algorithm to find the two largest numbers in a vector with k elements.

```
v = {x_1, x_2, x_3, ..., x_k}
first = second = -∞
for i = 1 to k do
        current = v(i)
        if first < current then
                second = first
                first = current
        else if second < current then
                second = current
        end if
end for
```

For non-degenerated transportation problems, the total number of assignments performed to solve the problem is always equal to $s + d - 1$ (or $n - 1$ when $s = d$ and $n = s + d$).

3.7.3a Worst-case scenario $\boldsymbol{O(f(n))}$ for VAM

Recall that $s = d$ and $n = s + d$. Considering the first assignment for VAM:

$$\text{time to compute penalties}$$
$$= \begin{cases} \text{each single row} & \rightarrow \; s = \dfrac{n}{2} \\ \text{each single column} & \rightarrow \; d = \dfrac{n}{2} \end{cases} \quad \text{(see pseudocode 1)}$$

$$\text{time to compute all penalties} = \frac{n}{2} \times \frac{n}{2} + \frac{n}{2} \times \frac{n}{2} = \frac{n^2}{2}$$

Assuming the worst-case scenario, after each assignment only one supplier (row of the cost matrix) or demand (column of the cost matrix) is eliminated – depending on the smallest value between quantity demanded and supply capacity. For the next assignment, the problem size is then reduced from n to $n - 1$ and the assignment process is repeated on the remaining reduced cost matrix. Therefore, the time to complete all $n - 1$ assignments is:

$$\sum_{k=1}^{n-1} \frac{k^2}{2} = \left(\sum_{k=1}^{n} \frac{k^2}{2}\right) - \frac{n^2}{2} = \left(\frac{n\,(n+1)(2n+1)}{12}\right) - \frac{n^2}{2} \Leftrightarrow \boldsymbol{O(n^3)}$$

The slope of $\log(n)$ *vs* $\log(\text{time complexity})$ is $\frac{\log(n^3)}{\log(n)} = 3$.

3.7.3b Average time complexity $\boldsymbol{\theta(f(n))}$ for VAM

Note that $O(n^3)$ is an upper bound for the time complexity of VAM. In practice, a single assignment can simultaneously eliminate both a demand point and its supplier from the cost matrix. This will happen every time that the remaining capacity of supply is equal to the demand quantity required at the point of assignment and it is expected to happen multiple times during the execution of VAM. Consequently, on average, some assignments may reduce the size of the cost matrix by 2 (one row and one column simultaneously) unlike the aforementioned worst-case scenario. For instance, for a giving assignment from supply i to demand j, if $s_i = d_i$, both the respective supply i (row) to demand j (column) will be removed from the cost matrix and the problem size reduces from say n to $n-2$ in one assignment.

For the worst-case scenario, the time to compute the penalties in a row or column is linear $O\left(\frac{n}{2}\right)$. In practice, because of the above-mentioned randomly occurring simultaneous reductions of rows and columns from the cost matrix, it is reasonable to estimate the actual time complexity to compute the penalties in rows and columns to be of sublinear time complexity, for example, $O\left(\log\left(\frac{n}{2}\right)\right)$.

Considering the first assignment for VAM:

$$\text{time to compute penalties} = \begin{cases} \text{each single row} & \rightarrow \log(s) = \log\left(\frac{n}{2}\right) \\ \text{each single column} & \rightarrow \log(d) = \log\left(\frac{n}{2}\right) \end{cases}$$

$$\text{time to compute all penalties} = \frac{n}{2} \times \log\left(\frac{n}{2}\right) + \frac{n}{2} \times \log\left(\frac{n}{2}\right) = n\log(n) - n\log(2)$$

Hence, the time to complete all $n-1$ assignments is:

$$\sum_{k=1}^{n-1} (n\log(n) - n\log(2)) = \left(\sum_{k=1}^{n-1} n\log(n)\right) - \left(\log(2)\sum_{k=1}^{n-1} n\right)$$

The time complexity of the second summation term is clearly $O(n^2)$ because:

$$\log(2)\sum_{k=1}^{n-1} n = \log(2)\frac{n\,(n+1)}{2} - n\,\log(2) \Leftrightarrow O(n^2)$$

The time complexity of the first summation term is:

$$\sum_{k=1}^{n-1} n\log(n) \propto \sum_{k=1}^{n} n\log(n) = \log(1^1 \times 2^2 \times 3^3 \times \ldots \times n^n) = \log\left(\prod_{k=1}^{n} k^k\right)$$

where:

$$\prod_{k=1}^{n} k^k = H(n) \propto A\,n^{(6n^2+6n+1)/12}\,e^{-n^2/4}$$

is the hyperfactorial of a positive integer n (Kinkelin, 1860) and $A \approx 1.28243$ the Glaisher–Kinkelin constant A.

$$\log(H(n)) \propto \log\left(A\,n^{(6n^2+6n+1)/12}\,e^{-n^2/4}\right)$$

$$= \log(A) + \left(\frac{6n^2+6n+1}{12}\right)\log(n) - \frac{n^2}{4}\log(e) \Leftrightarrow$$

$$\Leftrightarrow \boldsymbol{O(n^2\log(n))}$$

Therefore, the estimated average time complexity of VAM is $\theta\big(f(n)\big) = \theta(n^2\log(n))$ and the slope of $\log(n)$ *vs* log(time complexity) is:

$$\frac{\log(n^2\log(n))}{\log(n)} = \frac{2\log(n) + \log(\log(n))}{\log(n)} = \frac{\log(\log(n))}{\log(n)} + 2$$

Comparing this theoretical time complexity with our computational simulations, $\theta(n^2\log(n))$ produces effective results as shown in **Table 3**.

Table 3. CPU times in seconds for the VAM algorithm with the theoretical slope using natural $\log_e$ (base e).

n	VAM (CPU time in sec.)	(theoretical slope using $\log_e$): $\frac{\log_e(\log_e(n))}{\log_e(n)} + 2$
50	0.23	2.36
100	0.92	2.35
200	3.93	2.33
300	9.76	2.31
400	18.97	2.31
500	32.31	2.30
600	49.45	2.29
700	72.42	2.29
800	101.71	2.29
900	134.75	2.28
1000	175.62	2.28
1250	310.57	2.28
1500	493.83	2.28
2000	1046.90	2.27

These values will be compared with the experimental measurements made later in the thesis and shown in Figure 4, Page 84.

3.8 Optimal Solution

In the context of the proposed method, the optimal solution occurs when the total cost of transportation is minimized, that is where no other initial basic feasible solution produces a better total transportation cost (Khatun, 2012; Ishaq Abu Halawa et al., 2016). However, it should be noted that there are certain instances where the initial basic feasible solution is the same cost as the optimal solution. The Stepping-Stone and the Modified Distribution Mothed (MODI) are two well-known ways of verifying the optimality of an IBFS. The actual calculation will be shown in the following chapter to demonstrate the proposed method (TOCM-TBVAM) on a typical transportation problem where ties occur at multiple levels.

3.8.1 Testing for Optimality

There are two ways to test whether an IBFS is optimal, they are Stepping-Stone and the Modified Distribution Method. The following chapter will look at MODI to determine whether the IBFS obtained with the proposed method is optimal.

3.8.1a Stepping-Stone Method

This method is used to determine the optimality of the IBFS of NWCM, LCM, or VAM. This method is derived from the analogy of crossing a pond using Stepping-Stones, assuming that the transportation table is a pond, and the occupied cells are the stones needed to make the required movement within the stone (Winston and Goldberg, 2004; Ary and Herman, 2013). The main reason behind this is to check whether a transportation route that is not being used (i.e., an empty cell) would give a much lower total cost when used. In such an instance that the route is possible, then goods are allocated as much as possible to it until all requirements are satisfied.

3.8.1b Modified Distribution Method (MODI)

The MODI approach, also known as the U-V method or Simplex Method, checks whether a transportation problem solution is optimal by computing an improvement index for the unused square without sketching the entire closed path (Winston & Goldberg, 2004). As a result, when compared to other methods, the MODI method gives the best solution in the shortest amount of time (Ary & Herman, 2013; Limbore & Sachin Chandrakant, 2013). However, it should be noted that obtaining an IBFS before the application of MODI reduces the amount of time required to find the optimal solution.

3.9 Summary

This chapter presented the mathematical formulations of transportation problems, the balanced and unbalanced transportation problems, and the three main methods of finding the IBFS, namely, NWCM, LCM, and VAM, with VAM being the most efficient of these three methods as it gives a better IBFS, despite having a longer iteration process, which is compensated by a better IBFS and saving the time required to reach the Optimal solution. Additionally, the Computational time complexities of these main methods of finding the IBFS were discussed along with the two methods of testing for optimality: the Stepping-Stone and MODI. As a result, the next section will describe TBVAM and a newly proposed method TOCM-TBVAM, which uses the total opportunity cost matrix before the application of TBVAM on integer cost transportation problems.

CHAPTER FOUR

The Proposed Method (TOCM-TBVAM)

4.1 Introduction

In the previous chapter, we discussed the mathematical formulation of transportation problems, the two types of transportation problems, the three most widely used methods for obtaining the IBFS, and how to test for optimality using the Stepping-Stone and MODI. This chapter will discuss TBVAM and the more recent proposed method TOCM-TBVAM which are both methods of finding the IBFS of transportation problems. Since TBVAM is not as efficient as the state-of-the-art method as highlighted at the initial stage of this research, TOCM-TBVAM will be the focus in this chapter. The steps and pseudocode will therefore be presented, and two examples of transportation problems with integer cost values will be explored. One of these transportation problems would not include ties, and therefore performs exactly like TOCM-VAM in terms of the IBFS it obtained, while the other transportation problem will include breaking ties at several levels and the results on 35 benchmark transportation problems will be presented. It should be noted that these proposed algorithms work on balanced transportation problems.

4.2 TBVAM

Procedure to compute **TBVAM:**

1) Find the penalty cost from each row and column. This is done by subtracting the two minimum costs.
    - If two minimum costs are the same, the penalty cost is equal to zero.
2) Find the maximum penalty cost from each row and column.

3) Make maximum allocation to the cost cell with minimum cost in the maximum penalty cost.
4) If maximum penalty costs are the same, allocate the minimum cost in those cells.
5) If minimum costs are the same, make an allocation to the maximum mean cost in those cells.
6) If maximum mean costs are the same, make allocations to the maximum demand quantity that can be satisfied.
7) If the maximum demand quantity is the same, make allocations arbitrarily.
8) Repeat this iteration until all demand points have been met and supply points are exhausted.
9) Calculate the IBFS

The next approach is a recently proposed method that modifies TBVAM to achieve a better IBFS because TBVAM is not as efficient as the state-of-the-art method, but simply an improvement over VAM because it systematically breaks ties at several levels during the iteration process.

4.3 TOCM-TBVAM

The algorithm of TOCM-TBVAM works on getting the Total Opportunity Cost Matrix and then breaks ties during the iteration process.

Procedure to compute **TOCM-TBVAM:**

1) Subtract the minimum cost from each cost cell in the rows.

2) Subtract the minimum cost from each cost cell in the columns.

3) Add the corresponding cost from each row and column within each cost cell.

4) This gives the Total Opportunity Cost Matrix (TOCM).

5) Find the penalty cost from each row and column. This is done by subtracting the two minimum costs.

- If two minimum costs are the same, the penalty cost is equal to zero.
- In the case of real-valued cost, a tie is assigned if the two costs are within a set percentage of each other.

6) Find the maximum penalty cost from each row and column.

7) Make maximum allocations to the cost cell with minimum cost in the maximum penalty cost.

8) If maximum penalty costs are the same, make allocations to the minimum cost in those cells.

9) If minimum costs are the same, make allocations to the maximum mean cost in those cells.

10) If maximum mean costs are the same, make allocations to the maximum demand quantity that can be satisfied.

11) If the maximum demand quantity is the same, make allocations arbitrarily.

12) Repeat this iteration, until all demand points have been met and supply points are exhausted.

13) Calculate the IBFS

4.4 TOCM-TBVAM Pseudocode

	START
1-	In the cost matrix **subtract** the **minimum cost** in each **row** from each cost in the **row**.
2-	In the cost matrix **subtract** the **minimum cost** in each **column** from each cost in the **column**.
3-	Then **add** these costs in the cost matrix to make a new cost matrix **(TOCM)**
4-	Using the **TOCM** to replace the **original cost matrix**, find the **two minimum** costs in each row and each column and subtract them to get the penalty for the **row** and **column,** respectively.
5-	If the **two minimum** are **equal**, then the penalty is **zero**.
6-	Then find the **maximum penalty** of the **row** and the **maximum penalty** of the **column.**
7-	With **TOCM-TBVAM-TH,** multiply the **maximum penalty** with the **required percentage threshold** to achieve the **threshold proximity** with a range of costs to be considered as **minimum cost**.
8-	Make maximum allocation to the **minimum cost cell** within the chosen maximum penalty from **either** the row or column.
9-	**If** there is a **tie** in the **maximum penalty**, make maximum allocation to the maximum penalty with the **minimum cost**.
10-	**If** there is a **tie** in the **minimum cost** with **maximum penalty**, maximum allocation is made to the **minimum cost cell** with **maximum mean**.
11-	**If** there is a **tie** in the **maximum mean** with the **minimum cost** that has the **maximum penalty**, make maximum allocation to the cost cell where **maximum demand** can be **satisfied.**
12-	**If** there is a **tie** in the **maximum demand** that can be **satisfied** where the **maximum mean** with the **minimum cost** in the **maximum penalty**, make allocation **arbitrarily**.
13-	**Repeat** step **4** until all demand and supply points have been exhausted.
14-	Calculate the **Transportation Cost** by **multiplying the original cost** in each cost cell by the allocations in that cost cell and **add** all results together.
	END

4.5 Time complexity of TOCM-TBVAM

The main difference between VAM and TOCM-TBVAM is that the latter, computes the TOCM, in addition to the penalties for rows and columns (like VAM), and computes the average cost for each row and column in the cost matrix.

4.5.1 Worst-case scenario $\boldsymbol{O(f(n))}$ for TOCM-TBVAM

Considering the first assignment for TOCM-TBVAM:

$$\text{time to compute penalties} = \begin{cases} \text{each single row} & \rightarrow\ s = \dfrac{n}{2} \\ \text{each single column} & \rightarrow\ d = \dfrac{n}{2} \end{cases}$$

$$\text{time to compute averages} = \begin{cases} \text{each single row} & \rightarrow\ 1 \\ \text{each single column} & \rightarrow\ 1 \end{cases}$$

$$\text{time to compute all penalties and averages} = \frac{n}{2} \times \frac{n}{2} + \frac{n}{2} \times \frac{n}{2} + \frac{n}{2} \times 1 + \frac{n}{2} \times 1$$

$$= \frac{n^2}{2} + n$$

Assuming the worst-case scenario, the time to complete all $n-1$ assignments is:

$$\sum_{k=1}^{n-1} \frac{k^2}{2} + k = \left(\sum_{k=1}^{n} \frac{k^2}{2} \right) - \frac{n^2}{2} + \left(\sum_{k=1}^{n} k \right) - k$$

$$= \left(\frac{n\,(n+1)(2n+1)}{12} \right) - \frac{n^2}{2} + \left(\frac{n\,(n+1)}{2} \right) - n \Leftrightarrow$$

$$\Leftrightarrow \boldsymbol{O(n^3)}$$

Like for VAM, the slope of the worst-case scenario time complexity for TOCM-TBVAM is $\log(n)$ *vs* $\log(\text{time complexity})$ is $\frac{\log(n^3)}{\log(n)} = 3$. Thus, the time complexity of VAM and TOCM-TBVAM is comparable and both scale as a function of $O(n^3)$ as n increases.

4.5.2 Average time complexity $\boldsymbol{\theta(f(n))}$ for TOCM-TBVAM

Note that $O(n^3)$ is an upper bound for the time complexity of TOCM-TBVAM. Both VAM and TOCM-TBVAM follow the same assignment procedure. Analogously to

VAM, for TOCM-TBVAM, in practice, a single assignment can also simultaneously eliminate both a demand point and its supplier from the cost matrix. Therefore, using the same assumption of sublinear time complexity to compute the penalties of rows and columns, consider the first assignment for TOCM-TBVAM:

$$\text{time to compute penalties} = \begin{cases} \text{each single row} & \rightarrow \log(s) = \log\left(\frac{n}{2}\right) \\ \text{each single column} & \rightarrow \log(d) = \log\left(\frac{n}{2}\right) \end{cases}$$

$$\text{time to compute averages} = \begin{cases} \text{each single row} & \rightarrow 1 \\ \text{each single column} & \rightarrow 1 \end{cases}$$

$$\text{time to compute all penalties and averages} = \frac{n}{2} \times \log\left(\frac{n}{2}\right) + \frac{n}{2} \times \log\left(\frac{n}{2}\right) + \frac{n}{2} + \frac{n}{2}$$

$$= n \log\left(\frac{n}{2}\right) + n$$

$$\text{time to compute all penalties and averages} = n \log(n) - n \log(2) + n$$

Hence, the time to complete all $n - 1$ assignments is:

$$\sum_{k=1}^{n-1} (n \log(n) - n \log(2) + n) = \left(\sum_{k=1}^{n-1} n \log(n)\right) - \left(\log(2) \sum_{k=1}^{n-1} n\right) + \left(\sum_{k=1}^{n-1} n\right)$$

$$\Leftrightarrow \boldsymbol{O}(\boldsymbol{n}^2 \log(\boldsymbol{n}))$$

In conclusion, the estimated average time complexity of TOCM-TBVAM is $\theta(f(n)) = \theta(n^2 \log(n))$ and the average time complexity of both algorithms is comparable as each scale as a function of $\theta(n^2 \log(n))$ as n increases.

4.6 Advantages and disadvantages of TOCM-TBVAM

The concept of using the proposed algorithm TOCM-TBVAM to systematically break ties at severally levels to improve the initial basic feasible solution of the state-of-the-art method TOCM-VAM is a new algorithm in contrast to breaking ties arbitrarily. This provides TOCM-TBVAM with the advantage of being an efficient method for finding the initial basic feasible solution, which is sometimes close to or the same as the optimal solution. Another advantage of this proposed method is that the improvement of the initial basic feasible solution over the state-of-the-art algorithm would reduce the time required to achieve the optimal solution, as businesses would see this as a viable and more robust algorithm than TOCM-VAM, which would provide a better initial basic feasible solution that has been tested on large size transportation problems, saving the company time and money as it requires less time to achieve the optimal solution. In addition to the benefits of TOCM-TBVAM, an extension algorithm known as TOCM-TBVAM-TH is proposed to induce ties by using a percentage threshold in the maximum penalty as an indicative of ties in the minimum cost of both integer cost and real value cost, this gives the algorithm alternative pathways to access more costs that can be considered as minimum cost in the decision-making process. Furthermore, the systematic method of breaking ties used by this algorithm is simple to understand and can be applied to existing algorithms that break ties arbitrarily or those that require additional levels of ties breaking in the decision-making process to improve on the initial basic feasible solutions. This proposed algorithm, however, has a limitation in that it is only intended to perform on balanced transportation problems.

4.7 Numerical examples with illustration using TOCM-TBVAM

We illustrate TOCM-TBVAM by considering two typical transportation problems, one that does not have ties (Example 1) and one that has ties (Example 2).

4.7.1 Example 1

A company manufactures electrical generators and has three factories S1, S2, and S3 whose weekly production capacities are 15, 25, and 10 pieces of electrical generators, respectively. The company wants to supply electrical generators to its four retail shops located at D1, D2, D3, and D4 whose weekly demands are 5, 15, 15, and 15 pieces of electrical generators, respectively. The unit costs are displayed in **Table 4**.

Find the cost of transportation using TOCM-TBVAM.

4.7.1.a Solution to Example 1

The first step is to ensure that the transportation problem is balanced. Since the quantities supplied (50) is equal to the quantity demanded (50), it is a balanced transportation problem.

Table 4. Data from Example 1 shows the cost, demand, and supply matrix.

	Destinations				
Source	D1	D2	D3	D4	Supply
S1	10	2	20	11	15
S2	12	7	9	20	25
S3	4	14	16	18	10
Demand	5	15	15	15	

Iteration 1: We start by subtracting the minimum cost from each row in the cost matrix. For example, in **Table 4**, the minimum cost in the first row is 2, so we subtract 2 from each cost in row **S1**.i.e, (10 -2 = 8, 2 - 2 = 0, 20 – 2 = 18, 11 - 2 = 9). We also do the same for rows **S2** and **S3** which has the minimum cost of 7 and 4 respectively and subtract the minimum cost of 7 from rows **S2** and 4 from row **S3** and the result is shown in **Table 5**.

Iteration 2: From **Table 4**, we look for the minimum cost in each of the columns and subtract each column's minimum cost from each column. For example, column **D1**

has a minimum cost of 4, so we subtract 4 from column **D1. I.e.,** (10 - 4 = 6, 12 - 4 = 8, 4 - 4 = 0). We then do the same with each minimum cost in each column by subtracting 2, 9, and 11 from columns **D2**, **D3**, and **D4** respectively, and the result is shown in **Table 6**.

Iteration 3: We add the cost cells of **Table 5** and **Table 6** to form the **TOCM** in **Table 7**. For example, the first column **D1** has costs of 14, 13, and 0 because we have added costs from column **D1** of **Table 5** to column **D1** of **Table 6**. i.e., (8 + 6 = 14, 5 + 8 = 13, 4 + 0 = 4). We do the same for each column by adding the cost in **Table 5** and **Table 6**. This forms **Table 7** and will be used for the rest of the allocation process.

Table 5. Formation of opportunity cost matrix by subtracting the minimum cost in each row from each row.

	Destinations				
Source	D1	D2	D3	D4	Supply
S1	8	0	18	9	15
S2	5	0	2	13	25
S3	0	10	12	14	10
Demand	5	15	15	15	

Table 6. Formation of opportunity cost matrix by subtracting the minimum cost in each column from each column.

	Destinations				
Source	D1	D2	D3	D4	Supply
S1	6	0	11	0	15
S2	8	5	0	9	25
S3	0	12	7	7	10
Demand	5	15	15	15	

Table 7. The addition of the opportunity cost in Table 3 and Table 4 forms this Total Opportunity Cost Matrix.

	Destinations				
Source	D1	D2	D3	D4	Supply
S1	14	0	29	9	15
S2	13	5	2	22	25
S3	0	22	19	21	10
Demand	5	15	15	15	

Iteration 4: we find the two minimum costs from each row and each column and subtract the smallest cost from the next smallest cost to get a cost known as the penalty cost. In case of having two smallest costs as the same values or cost, the penalty will be 0. In **Table 8**, the penalty for **D1** is 13 because we have subtracted 0 from 13. Likewise in column **D2**, **D3** and **D4**, we do 5 – 0 = 5, 19 - 2 = 17, 21 – 9 = 12, respectively. We also do the same for the rows i.e., **S1** = 9 - 0 = 9, **S2** = 5 - 2 = 3, and **S3** = 19 - 0 = 19. We then look for the maximum penalty which is 19 and it is in row **S3**. Along that row, we look for the minimum cost and make the maximum possible allocations to that cost cell. The minimum cost in row **S3** is 0, so we make the maximum possible allocations according to the demand and supply constraints which is 5, denoted by **0[5]** in **Table 8**. Find the smallest cost in row **S3**, which is 0, and make the maximum possible allocations. Here 5 is the maximum possible allocation because of the demand constraint and what is left in **S3** supply of 10 is 5, denoted as **10|5**. Once the demands are met or the supply is exhausted, we cross out the column or row. In this case, we cross out **D1** because demand has been met shown in **Table 8**.

Table 8. TOCM with the first allocation to cell **S3D1**.

	Destinations					
Source	D1	D2	D3	D4	Supply	Row Penalty Cost
S1	14	0	29	9	15	9
S2	13	5	2	22	25	3
S3	**0[5]**	22	19	21	10|5	19
Demand	5|0	15	15	15		
Column Penalty Cost	13	5	17	12		

Iteration 5: With **D1** demand satisfied, we again subtract the smallest of the minimum cost from the next minimum cost of each row and column as before to form the row and column penalty cost. We then look for the maximum penalty cost which is 17 in column **D3** and look for the minimum cost in that column to make allocations. The minimum cost is 2, we, therefore, make allocations (15 units) to the cell **S2D3** and adjust both demand and supply accordingly as shown in **Table 9.**

Table 9. TOCM with the second allocation to cell **S2D3**.

	Destinations					
Source	D1	D2	D3	D4	Supply	Row Penalty Cost
S1	-	0	29	9	15	9
S2	-	5	**2[15]**	22	25|10	3
S3	[5]	22	19	21	5	2
Demand	-	15	15|0	15		
Column Penalty Cost	-	5	17	12		

Iteration 6: with column **D3** fully met, we do as before in columns **D2** and **D4** to find the penalty (**D2 = 5**, **D4 =12**), and likewise with rows **S1 = 9, S2 = 17**, and **S3 = 1**. Since 17 is the maximum penalty, will look at the minimum cost in row **S2**. The minimum cost is 5 and we make a maximum allocation of 10 to cell **S2D2 and** cross out row **S2** since the supply is fully exhausted as shown in **Table 10.**

Table 10. TOCM with the third allocation in cell **S2D2**.

	Destinations					
Source	D1	D2	D3	D4	Supply	Row Penalty Cost
S1	-	0	-	9	15	9
S2	-	**5[10]**	[15]	22	10|0	17
S3	[5]	22	-	21	5	1
Demand	-	15	-	15		
Column Penalty Cost	-	5	-	12		

Iteration 7: In **Table 11** we have rows **S1** and **S3** with penalty costs of 9 and 1, respectively. Likewise in columns **D2** and **D4**, we have penalty costs of 22 and 12, respectively. Since **D2** has the maximum penalty, we look for the minimum cost in that column which is 0 and make maximum allocations possible of 5, and this ensures that **D2** demand is met as seen in **Table 11** and we cross out the column.

Table 11. TOCM with the fourth allocation to cell **S1D2.**

	Destinations					
Source	D1	D2	D3	D4	Supply	Row Penalty Cost
S1	-	**0[5]**	-	9	15|10	9
S2	-	[10]	[15]	-	-	-
S3	[5]	22	-	21	5	1
Demand	-	5|0	-	15		
Column Penalty Cost	-	22	-	12		

Iteration 8: With only column **D4** left to be satisfied, we then make allocations according to the demand and supply constraint. Therefore, cells **S1D4** and **S3D4** will have 10 and 5 units respectively as shown in **Table 12**.

Table 12. TOCM with the fourth allocation to cells **S1D4** and **S3D4**.

	Destinations					
Source	D1	D2	D3	D4	Supply	Row Penalty Cost
S1	-	[5]	-	**9[10]**	10	9
S2	-	[10]	[15]	-	-	-
S3	[5]	-	-	**21[5]**	5	1
Demand	-	-	-	15		
Column Penalty Cost	-	-	-	12		

Iteration 9: The occupied cells are then replaced with the original cost accordingly as shown in **Table 13**, this completes the decision-making process in this example.

Table 13. Final allocation table with the cost for each unit transported.

	Destinations				
Source	D1	D2	D3	D4	Supply
S1	-	2[5]	-	11[10]	-
S2	-	7[10]	9[15]	-	-
S3	4[5]	-	-	18[5]	-
Demand	-	-	-	-	

Total transportation cost = $(4 \times 5) + (2 \times 5) + (7 \times 10) + (9 \times 15) + (11 \times 10) + (18 \times 5) = \mathbf{435}$

TOCM-TBVAM = **435**

TOCM-VAM = 435 (Previously computed)

Optimal Solution = 435 (Previously computed)

In this first example, both TOCM-VAM and TOCM-TBVAM produced the same IBFS as there were no ties throughout the iteration process. As a result, TOCM-TBVAM would function similarly to TOCM-VAM, except when there are ties either in the maximum penalty, minimum cost, or maximum mean cost and maximum quantity demanded.

4.7.2 Example 2

Using the proposed method TOCM-TBVAM, the following example will look at the effect and benefit of having and breaking ties at several levels.

Table 14. Data for Example 2 shows the cost, demand, and supply matrix.

	Destinations			
Source	D1	D2	D3	Supply
S1	1	1	1	12
S2	1	8	1	14
S3	1	1	1	16
Demand	8	22	12	

4.7.2a Solution to Example 2

Iteration 1: We start by subtracting the minimum cost from each row in the cost matrix. For example, in **Table 14**, the minimum cost in the first row is 1, so we subtract 1 from each cost in row **S1**.i.e, (1 - 1 = 0, 1 - 1 = 0, 1 – 1 = 0). We also do the same for rows **S2** and **S3** which has the minimum cost of 1 and 1 respectively and subtract this minimum cost of 1 from row **S2** and 1 from row **S3** and the result is shown in **Table 15**.

Iteration 2: From **Table 14**, we look for the minimum cost in each of the columns and subtract each column's minimum cost from each column. For example, column **D1** has a minimum cost of 1, so we subtract 1 from column **D1**.i.e, (1 - 1 = 0, 1 - 1 = 0, 1 - 1 = 0). We then do the same with each minimum cost in each column by subtracting 1 and 1 from columns **D2** and **D3** respectively, and the result is shown in **Table 16**.

Iteration 3: We add the cost cells of **Table 15** and **Table 16** to form the **TOCM** in **Table 16**. For example, the first column **D1** has costs 0, 0, and 0 because we have added costs from column **D1** of **Table 15** to column **D1** of **Table 16**. i.e., (0 + 0 = 0, 0 + 0 = 0, 0 + 0 = 0). We do the same for each column by adding the cost in **Table 15**

and **Table 16**. This forms TOCM as **Table 17** and it will be used for the rest of the allocation process.

Table 15. Formation of opportunity cost matrix by subtracting the minimum cost in each row from each row.

	Destinations			
Source	D1	D2	D3	Supply
S1	0	0	0	12
S2	0	7	0	14
S3	0	0	0	16
Demand	8	22	12	

Table 16. Formation of opportunity cost matrix by subtracting the minimum cost in each column from each column.

	Destinations			
Source	D1	D2	D3	Supply
S1	0	0	0	12
S2	0	7	0	14
S3	0	0	0	16
Demand	8	22	12	

Table 17. The addition of the opportunity cost in **Table 15** and **Table 16** forms this Total Opportunity Cost Matrix.

	Destinations			
Source	D1	D2	D3	Supply
S1	0	0	0	12
S2	0	14	0	14
S3	0	0	0	16
Demand	8	22	12	

Table 18. TOCM with the first allocation to cell **S3D2**.

	Destinations				
Source	D1	D2	D3	Supply	Row Penalty Cost
S1	0	0	0	12	0
S2	0	14	0	14	0
S3	0	**0[16]**	0	16|0	0
Demand	8	22|6	12		
Column Penalty Cost	0	0	0		

Iteration 4: Since all the penalty is = 0, the maximum penalty will be considered as 0, which is considered a tie. We then look for the minimum cost in these rows and columns, we also noticed that the minimum cost of 0 appears in all rows and columns, then the deciding factor is to calculate the mean cost to break the tie. The mean is equal to 0 in rows **S1** and **S3** and columns **D1** and **D3**. Column **D2** and row **S2** have the maximum mean which is approximately 4.7 (0 + 14 + 0 = 14 ÷ 3 = 4.66). This is also a tie, then the tie is broken by looking at which of these with the maximum mean have the maximum demand quantity that can be satisfied. So **S3D2** is allocated 16 units according to the demand and supply constraints, and we cross out **S3** since all supply is exhausted as shown in **Table 18.**

Table 19. TOCM with the second allocation to cell **S1D2**.

	Destinations				
Source	D1	D2	D3	Supply	Row Penalty Cost
S1	0	**0[6]**	0	12|6	0
S2	0	14	0	14	0
S3	-	[16]	-	-	-
Demand	8	6|0	12		
Column Penalty Cost	0	14	0		

Iteration 5: The maximum penalty is 14 which is in column **D2** (14 – 0 =14), and then make maximum allocations to the minimum cost cell, **S1D2**. We then adjust demand and supply quantities accordingly in **Table 19.**

Table 20. TOCM with the third allocation to cell **S2D3**.

	Destinations				
Source	D1	D2	D3	Supply	Row Penalty Cost
S1	0	[6]	0	6	0
S2	0	-	**0[12]**	14|2	0
S3	-	[16]	-	-	-
Demand	8	-	12|0		
Column Penalty Cost	0	-	0		

Iteration 6: since all the penalty costs are the same, and minimum costs are the same, and the mean costs are the same as shown in **Table 20**, allocation is made to the cost cell where maximum demand can be satisfied. i.e., **S2D3**. We make a maximum allocation of 12 units to cell **S2D3** and then cross out column **D3** since all demand is satisfied in that column**.**

Table 21. TOCM with the fourth allocations to cells **S1D1** and **S2D1**.

	Destinations				
Source	D1	D2	D3	Supply	Row Penalty Cost
S1	**0[6]**	[6]	0	6|0	0
S2	**0[2]**	-	[12]	2|0	0
S3	-	[16]	-	-	-
Demand	8|0	-	-		
Column Penalty Cost	0	-	-		

Iteration 7: Since **D1** is the only unmet column, allocation is made accordingly in **Table 21**.

Iteration 8: The occupied cells are then replaced with the original cost accordingly as shown in **Table 22**, this completes the decision-making process in this example.

Table 22. Final allocation table with the cost for each unit transported.

	Destinations			
Source	D1	D2	D3	Supply
S1	1[6]	1[6]	-	-
S2	1[2]	-	1[12]	-
S3	-	1[16]	-	-
Demand	-	-	-	

Total transportation cost = (1 x 6) + (1 x 2) + (1 x 6) + (1 x 16) + (1 x 12) = **42**

TOCM-TBVAM = **42**

TOCM-VAM = **56** (Previously computed)

Optimal solution = **42** (Previously computed)

Example 2 shows that 42 is the optimal solution which differs from the IBFS of TOCM-VAM which is equal to 56.

The next step is to check for Optimality using MODI as discussed in Chapter Three.

4.8 Checking for Optimality using MODI in Example 2.

Using the Modified Distribution Method on the IBFS by TOCM-TBVAM, we consider the following steps. Using **Table 22** with the occupied cells of cost and quantity shipped from supply to demand points. We then follow these steps:

Step 1- We set up an equation for each occupied cost cell.

$$R_1 + K_1 = 1 \quad (13)$$

$$R_1 + K_2 = 1 \quad (14)$$

$$R_2 + K_1 = 1 \quad (15)$$

$$R_2 + K_3 = 1 \quad (16)$$

$$R_3 + K_2 = 1 \quad (17)$$

Step 2 - We compute the improvement index for each unused cost cell in **Table 22**.

Thus, let $R_1 = 0$, we can solve for K_1 , K_2 , K_3 , R_2 *and* R_3

$$0 + K_1 = 1, \qquad K_1 = 1 \quad (18)$$

$$0 + K_2 = 1, \qquad K_2 = 1 \quad (19)$$

$$R_2 + 1 = 1, \qquad R_2 = 0 \quad (20)$$

$$0 + K_3 = 1, \qquad K_3 = 0 \quad (21)$$

$$R_3 + 1 = 1, \qquad R_3 = 0 \quad (22)$$

Step 3 - We compute the improvement index for each unused cell in **Table 22.**

Thus,

Improvement index $=I_{ij} = C_{ij} - R_i - K_j$ (23)

$=I_{13} = C_{13} - R_1 - K_3 = 1 - 0 - 0 = 1$ (24)

$=I_{22} = C_{22} - R_2 - K_2 = 8 - 0 - 1 = 7$ (25)

$=I_{31} = C_{31} - R_3 - K_1 = 1 - 0 - 1 = 0$ (26)

$=I_{33} = C_{33} - R_3 - K_3 = 1 - 0 - 0 = 1$ (27)

Therefore, from the result of the improvement index for the unused cell not having a negative value, the IBFS of TOCM-TBVAM is optimal.

Table 23. Benchmark results for 35 transportation problems with integer values.

VAM	TBVAM	TOCM-VAM	TOCM-TBVAM	Best of TOCM-TBVAM (TOCM-TBVAM-TH)	OPTIMAL
5600	**5600**	**5600**	**5600**	**5600**	5600
1220	**1165**	**1165**	**1165**	**1165**	1160
68	**68**	**68**	**68**	**68**	68
390	**390**	**390**	**390**	**390**	390
355	**355**	**355**	**355**	**355**	355
114	114	**111**	**111**	**111**	111
199	199	**183**	**183**	**183**	183
2657000	**2657000**	2658000	2658000	2658000	2655600
92	92	**83**	**83**	**83**	83
1500	**1390**	1450	1450	**1390**	1390
859	859	**799**	**799**	**799**	799
955	955	**880**	**880**	**880**	880
285	**285**	**285**	**285**	**285**	285
322	**322**	**322**	**322**	**322**	322
290	**290**	**290**	**290**	**290**	290
779	779	**743**	**743**	**743**	743
112	**112**	**112**	**112**	**112**	112
1104	**1104**	**1104**	**1104**	**1104**	1102
2224	**2224**	**2224**	**2224**	**2224**	2202
116	**116**	**116**	**116**	**116**	112
2130	**2070**	2130	2130	2130	2070
1930	1930	**1900**	**1900**	**1900**	1900
59	**59**	**59**	**59**	**59**	59
2310	2220	**2170**	**2170**	**2170**	2170
21030	21030	**20550**	**20550**	**20550**	20550
470	**410**	470	470	440	410
750	**674**	**674**	**674**	**674**	674
56	**42**	56	**42**	**42**	42
3663	3663	**3513**	**3513**	**3513**	3513
28	**28**	**28**	**28**	**28**	28
475	475	**435**	**435**	**435**	435
80	80	**76**	**76**	**76**	76
150	**139**	145	145	145	139
849	849	**809**	**809**	**809**	809
465	465	**417**	**417**	**417**	417

The proposed algorithm TOCM-TBVAM, is compared with VAM, TBVAM, TOCM-VAM, and the optimal cost for 35 benchmark transportation problems with integer costs, from the published literature which can be seen in **Appendix 2** of this thesis, with the results shown in **Table 23.** It should be noted that because of the arbitrary allocation that occurs when there is a tie, TOCM-VAM does get different IBFS because ties do give different paths and therefore different IBFS (Hossain et al., 2020, Murugesan and Esakkiammal, 2020, Sultana et al., 2022). Therefore, the TOCM-VAM algorithm must be performed several times to get a better IBFS, which will increase the computational time for large transportation problems. In addition, to maintain consistency and accuracy in this research, computer-generated results are used throughout.

The results in **Table 23** shows that TOCM-VAM and TOCM-TBVAM generated the same IBFS cost except in one instance where TOCM-TBVAM outperformed TOCM-VAM because of the tie-breaking method of TOCM-TBVAM. This emphasises the significance of the tie-breaking method in integer cost transportation problems utilising the proposed TOCM-TBVAM algorithm. Although the proposed method includes the formation of TOCM before implementing TBVAM (which systematically breaks ties at multiple levels), it should be noted that TOCM-VAM has never outperformed TOCM-TBVAM. It should also be noted that VAM never outperformed any of the proposed algorithms. i.e., TBVAM, TOCM-TBVAM, and TOCM-TBVAM-TH.

Furthermore, to illustrate the benefit of inducing ties at specific percentage threshold intervals in integer cost transportation problems, which may also be applied to real value cost transportation problems, the extension of TOCM-TBVAM known as TOCM-TBVAM-TH shows a slight improvement on the proposed algorithm TOCM-TBVAM in two instances shown in **Table 23**, resulting in a more efficient solution that is closer to the optimal solution and another instance that is the same as the optimal solution.

The overall number of wins and the average proportional difference from the optimal cost are shown in **Table 24**.

In pairwise comparisons in **Table 24**, where a tie in the number of wins is considered as 0.5 because they share the win of 1 equally, TOCM-VAM had 14 more wins than TBVAM, but TOCM-TBVAM outperforms TOCM-VAM. When the number of wins for TOCM-VAM is compared to TOCM-TBVAM-TH, the algorithm that uses a percentage threshold in the maximum penalty to induce ties, the best of TOCM-TBVAM-TH has 4 more wins, indicating an improvement when different percentage thresholds are applied which is significant to getting the optimal solutions. In terms of proximity to optimal cost, the results revealed that the method that achieves the closest cost to optimal is TOCM-TBVAM-TH, and TOCM-TBVAM-TH with an average proportional difference to the optimal score of 0.44% whereas TOCM-VAM was on average 1.85% off, which is four times more showing the significant improvement of the extension of the proposed algorithm. The application of TOCM-TBVAM-TH will be discussed in more detail in Chapter 5.

Table 24. Wins and proportional difference from optimal cost for cost matrices with integer values. Half scores indicate ties.

TBVAM vs TOCM-VAM		All Best TOCM-TBVAM-TH vs. TOCM-VAM		Proportion diff from Optimal: TOCM-TBVAM-TH	Proportion diff from Optimal: TOCM-VAM
0.5	0.5	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.43%	0.43%
0.5	0.5	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
1	0	0.5	0.5	0.09%	0.09%
0	**1**	0.5	0.5	0.00%	0.00%
1	0	**1**	0	**0.00%**	4.32%
0	**1**	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.18%	0.18%
0.5	0.5	0.5	0.5	1.00%	1.00%
0.5	0.5	0.5	0.5	3.57%	3.57%
1	0	0.5	0.5	2.90%	2.90%
0	**1**	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
1	0	**1**	0	**7.32%**	14.63%
0.5	0.5	0.5	0.5	0.00%	0.00%
1	0	**1**	0	**0.00%**	33.33%
0	**1**	0.5	0.5	0.00%	0.00%
0.5	0.5	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
1	0	**1**	0	**0.00%**	4.32%
0	**1**	0.5	0.5	0.00%	0.00%
0	**1**	0.5	0.5	0.00%	0.00%
Total TBVAM	**Total TOCM-VAM**	**Total TOCM-TBVAM-TH**	**Total TOCM-VAM**	**Average for TOCM-TBVAM-TH**	**Average for TOCM-VAM**
13.5	**21.5**	**19.5**	15.5	**0.44%**	1.85%

4.9 Summary

The proposed approach TOCM-TBVAM was introduced in this chapter along with the pseudocode and the computational time complexity of TOCM-TBVAM. Furthermore, two examples were used to illustrate its similarities and differences from the state-of-the-art method TOCM-VAM in the context of breaking ties. **Table 23** compares the results with VAM, TBVAM, which was the initial proposed approach, TOCM-VAM, and TOCM-TBVAM. The comparison also includes the extension of TOCM-TBVAM known as TOCM-TBVAM-TH, which considers the threshold proximity on the maximum penalty that gives different pathways and takes advantage of breaking ties at several levels that improve the IBFS. Nevertheless, before comparing the results to TOCM-VAM in terms of real-valued cost, the next chapter will look at how and why two batches of 10,000 transportation problems were simulated and generated in this research.

CHAPTER FIVE

Data Description and Application of TOCM-TBVAM-TH

5.1 Introduction

The preceding Chapter evaluated the proposed method (TOCM-TBVAM), which considers total opportunity cost before making allocations in each iteration. It also demonstrated that when there is a tie, TOCM-TBVAM outperforms the state-of-the-art, TOCM-VAM in the transportation problems tested. TOCM-VAM makes arbitrary allocations in ties, resulting in the algorithm receiving a lower IBFS in some cases, and a higher IBFS in the same transportation problems with ties. The results also demonstrate an improvement in TBVAM, which was the starting point of the research algorithm. Also included is the result of TOCM-TBVAM-TH, which induces ties at the maximum penalty cost by giving different pathways and takes the benefit of systematically breaking ties to produce a lower IBFS. This method additionally indicates an improvement over the state-of-the-art method with integer cost and therefore should be applied to real-value costs. As a result, this Chapter will look at how the 20,000 transportation problems were generated with real-valued costs to evaluate the performance of both TOCM-VAM and TOCM-TBVAM-TH. The key reason for this is that the two batches of 10,000 simulated transportation problems will represent real-world transportation problems that have a large size dimension with real-valued costs, while also addressing a research gap because, according to published literature, most algorithms for finding IBFS have not been tested on a large set of transportation problems with large size dimensions and real-valued costs. In addition, this gives the findings in this research a statistically significant meaning substantiated by consistent and credible results.

5.2 Simulation of Transportation Problems

A set of two batches of 10,000 simulated transportation problems were generated with real-valued cost matrices to ascertain the effectiveness of the application of the proposed TOCM-TBVAM-TH compared with the state-of-the-art method TOCM-VAM. Apart from the fact that no data was accessible after writing letters to several logistic organisations, the purpose of the stimulation of these transportation problems was to mirror real-world cases of transportation problems. For instance, as indicated by the publications, using a real-valued cost matrix was also found to be a gap in the literature review. The use of a large-size dimension for both demand and supply points were also identified in the literature as a gap because most IBFS obtained in the publications were of small-size dimensions, i.e., 3x4 and 7x7. The simulated transportation problems were generated uniformly at random with continuous cost values to reflect real-world cost values for which exact ties are less likely.

To generate the first set of 2,000 simulated transportation problems, the minimum cost values and maximum cost values were set at 10% below and 10% above a fixed cost (100) accordingly. For instance, the minimum cost for the first set of 2,000 simulated transportation problems was 90, while the maximum cost was 110. The fixed cost for the following 2,000 simulated problems was set at 20% below and 20% above 100, making the minimum cost as 80 and the maximum cost as 120. This process is repeated at 30% and 40% until the fixed cost of 100 is 50% below and 50% above, in other words, the randomly generated cost in these 2,000 simulated transportation problems will fall between a minimum cost range of 50 and a maximum cost range of 150. We have used the range of 50% below and 50% above the fixed cost value because a real-world cost value for a unit good should be within that range. Another set of 10,000 simulated transportation problems is then generated and this forms 2 different lots of 5 sets of 2,000 transportation problems making a total of 20,000 simulated transportation problems. With these generated simulated transportation problems, it is expected that real product costs will not deviate too much from a fixed cost, therefore using 100 as the fixed cost or any other value will not have any impact because the fixed cost is scale-invariant. The minimum and maximum number of the supply points are 10 and 100, while the minimum and maximum number of the demand points are 50 and 200, respectively. The supply and demand points denote the number of sources and the number of destinations, respectively. Similarly, the minimum demand quantity is set at 100 and the maximum demand quantity is set at 1000, so the simulation randomly generated the demand quantity within this range. The minimum supply quantity is set at 500 and the maximum supply quantity is set at 5000, therefore the generated simulated quantity of supply was within this range. In all the simulated transportation problems, the sum of the quantities supplied is equal to the sum of the quantities demanded making it a balanced transportation problem. Please, see the R code in **Appendix 1** for the simulations, and the benchmark datasets along with the 20,000 simulated transportation problems in **Appendix 2,** which are available for download.

5.3 A step-by-step guide on how the 20,000 transportation problems were generated

An explanation of each step used to create the 20,000 balanced transportation problems with real-valued costs using the R Code provided in Appendix 1.

Guide to how the first set of 2,000 simulated transportation problems with real-valued cost was generated.

```r
## R Code for Simulated Transportation Problems with Continuous Cost
#Clears the memory
rm(list=ls(all=TRUE))

library(openxlsx)
### nprobs denotes the number of transportation problems that needs to be simulated
nProbs = 2000
### rangeDemand denotes the  minimum and maximum  demand quantities.
rangeDemand = c(100, 1000)
#### rangeSupply denotes the  minimum and maximum  supply quantities.
rangeSupply = c(500, 5000)
###### nDem denotes the minimum and maximum demands points
nDem = c(50, 200)
### nSup denotes the minimum and maximum supply points
nSup = c(10, 100)
### rangeCost denotes the minimum and maximum cost range for the simulated TP
rangeCost = c(90, 110)
##### costChange controls the variation(min & max) values of random costs around a randomly draw
costChange = 10
```

Here nProbs is set to 2000 to generate the first 2000 transportation problems with real-valued cost.

We set the cost range 10% below the fixed cost of 100 and 10% above the fixed cost of 100.

To simulate the next 2000 transportation problems, we retain nProbs at 2000 because we are generating another 2000 to make 4000 simulated transportation problems, we just extend the cost range to be 20% above and 20% below the fixed cost of 100. For the next 2000 transportation problems to make 6000 simulated transportation problems, we keep nProbs as 2000 and extend the cost range to 30% above and 30% below the fixed cost of 100. Similarly, to generate the next 2000 transportation problems to make 8000 simulated transportation problems with nProbs as 2000, we extend the cost range to be 40% above and 40% below the fixed cost of 100. Finally, to generate the last 2000 transportation problems to make 10,000 simulated transportation problems with nProbs still at 2000, we extend the cost range to be 50% above and 50% below the fixed cost of 100. This completes the procedure of creating the first 10,000 simulated transportation problems and this process can be repeated to create the remaining 10,000 transportation problems resulting in a total of 20,000 simulated transportation problems for the experiment.

Table 25. An example of a simulated balanced transportation problem with 10 supply points and 8 demand points.

	D1	D2	D3	D4	D5	D6	D7	D8	Supply
S1	84.47	84.87	85.23	81.28	85.57	83.91	84	82.78	112
S2	81.75	82.37	85.96	83.21	83.17	79.25	79.73	80.98	746
S3	79.64	82.85	82.01	81.04	85.29	78.59	85.61	83.18	755
S4	84.49	82.45	81.66	80.65	83.3	79.58	83.93	83.72	742
S5	78.23	80.66	80.63	82.59	83.36	85.23	81.59	81.09	654
S6	80.36	81.1	81.23	82.85	85.09	82.01	80.23	85.65	601
S7	85.02	85.61	84.61	79.52	78.05	84.13	81.99	84.78	673
S8	80.95	85.73	82.59	84.53	85.9	81.58	83.21	84.84	542
S9	78.82	85.49	78.03	81.75	78.96	78.04	85.09	78.4	891
S10	81.55	78.41	82.54	82.35	80.22	79.22	82.91	79.49	635
Demand	554	944	983	855	834	702	919	560	6351

Table 25 shows a simulated balanced transportation problem that was generated with the R software tool showing real-valued costs. However, this is a small-dimension transportation problem when compared to the size of the transportation problems used in this research which are available for download (see **Appendix 1** and **Appendix 2**).

5.4 Application of TOCM-VAM and TOCM-TBVAM-TH on the simulated transportation problem with real-valued costs.

A set of 20,000 simulated transportation problems was generated with real-valued cost matrices to ascertain the effectiveness of the application of the proposed TOCM-TBVAM-TH compared with the state-of-the-art method TOCM-VAM. Ties are induced at multiple proximity thresholds at 10% intervals. Do note that TOCM-TBVAM never loses compared with TOCM-VAM because while TOCM-VAM works on arbitrary allocation when there is a tie in maximum penalty cost and minimum cost, TOCM-TBVAM on average improves the IBFS by way of systematically breaking ties. Therefore, for TOCM-VAM to have a better IBFS, it will have to run the algorithm multiple times to get a chance at the required improvement, and the TOCM-VAM-TH solution corresponds to an extreme of the values of the interval width to declare a tie. Both methods come at a computational cost that is discussed in the conclusion chapter.

The 20,000 simulated transportation problems were produced randomly and independently. They were then analysed in sets of 1000 transportation problems to estimate the variation due to chance. For this reason, the means of each set of 1000 were calculated along with the means of the percentage improvement and the standard deviation in these sets are reported in **Table 26**.

Table 26 shows the number of wins for the twenty sets of 1,000 simulated transportation problems when TOCM-TBVAM-TH is compared with TOCM-VAM. For example, set 1000A comprises 1000 simulated transportation problems resulting in 22 wins for TOCM-TBVAM-TH over TOCM-VAM. In this set, the percentage difference of TOCM-TBVAM-TH to the optimal cost is 10.40% while that of TOCM-VAM is higher (12.83%), resulting in about 2.43% improvement on TOCM-VAM by the application of ties in maximum penalty with TOCM-TBVAM-TH. Therefore, on average, the results indicate that there is a 2% improvement on TOCM-VAM when TOCM-TBVAM-TH is used on real value cost transportation problems.

Table 26. Number of wins from twenty batches of 1000 simulated transportation problems with real-valued cost matrices, using multiple proximity thresholds in TOCM-TBVAM-TH.

Set of 1000 Transportation Problems	Number of wins of TOCM-TBVAM-TH	TOCM-TBVAM-TH percentage difference to Optimal cost	TOCM-VAM percentage difference to Optimal cost	Percentage improvement
1000A	22	**10.40%**	12.83%	**2.43%**
1000B	17	**7.49%**	9.74%	**2.25%**
1000C	27	**11.51%**	13.75%	**2.25%**
1000D	24	**11.08%**	13.73%	**2.65%**
1000E	27	**11.80%**	14.54%	**2.74%**
1000F	19	**9.69%**	11.83%	**2.14%**
1000G	23	**11.46%**	14.71%	**3.24%**
1000H	20	**9.75%**	11.75%	**1.99%**
1000I	23	**10.75%.**	13.07%	**2.32%**
1000J	19	**7.75%**	9.70%	**1.95%**
1000K	21	**9.94%**	11.37%	**1.43%**
1000M	21	**10.48%**	12.84%	**2.36%**
1000N	20	**9.30%**	10.75%	**1.45%**
1000O	25	**12.38%**	13.79%	**1.41%**
1000P	23	**10.36%**	12.41%	**2.05%**
1000Q	22	**8.49%**	10.27%	**1.77%**
1000R	18	**9.13%**	10.27%	**1.44%**
1000S	24	**10.78%**	12.86%	**2.08%**
1000T	28	**13.00%**	14.80%	**1.81%**
1000U	20	**9.96%**	11.79%	**1.83%**
Average	22.15	**10.18%**	12.34%	**2.08%**
Standard Deviation	3.07	1.43%	1.63%	0.43%

5.5 Summary

This chapter explained and provided a step-by-step guide on how the 20,000 simulated transportation problems were generated, which is to reflect the real-world transportation problems with real value cost and large size dimensions. These were noted as the research gap and motivation for the research in the literature review. These transportation problems were also used to compare the performance of TOCM-TBVAM-TH to the state-of-the-art method TOCM-VAM. The multiple pathways created by TOCM-TBVAM-TH, which is the key factor in this proposed method, take advantage of inducing ties at maximum penalty so that more real value cost could be regarded as minimum cost in the sample space during the iteration process. These IBFS obtained in the simulated transportation problems have shown that TOCM-TBVAM-TH can improve on the state-of-the-art method TOCM-VAM by providing a more efficient way of inducing ties and then breaking ties at several levels during the iteration process of the algorithm to provide a lower IBFS that saves time to get the optimal solution. The following chapter will discuss the findings and results of the comparison between TOCM-TBVAM-TH and TOCM-VAM.

CHAPTER SIX

Results and Discussion

6.1 Introduction

The previous chapter presented TOCM-TBVAM-TH applications on the simulated transportation problems generated, which are available for download (see **Appendix 2**). The generated datasets were of real-valued costs and since it was extremely difficult to have ties in real-valued costs transportation problems, percentage threshold proximity was applied using TOCM-TBVAM-TH to induce ties and that gives the algorithm different pathways during the iteration process to have more alternative solutions to choose from so that the next iteration will not make allocations to a higher cost. This application of TOCM-TBVAM-TH has shown that on average it can improve on the state-of-the-art method, TOCM-VAM by providing a more efficient and lower IBFS that would in turn make the process of obtaining the optimal solution much faster, this can potentially save companies money and time. Therefore, whatever the situation may be, TOCM-TBVAM-TH is an improvement on TOCM-VAM by inducing ties and systematically breaking ties. This chapter will present the statistical analysis of the results, and computational complexity and provide the basis for discussion.

6.2 Findings

The results obtained in this research are outlined as follows.

6.2.1 Statistical Analysis

To start a statistical analysis of the results of TOCM-VAM and the best of TOCM-TBVAM-TH with real-valued costs, we require a graphical representation of the findings in the form of a histogram.

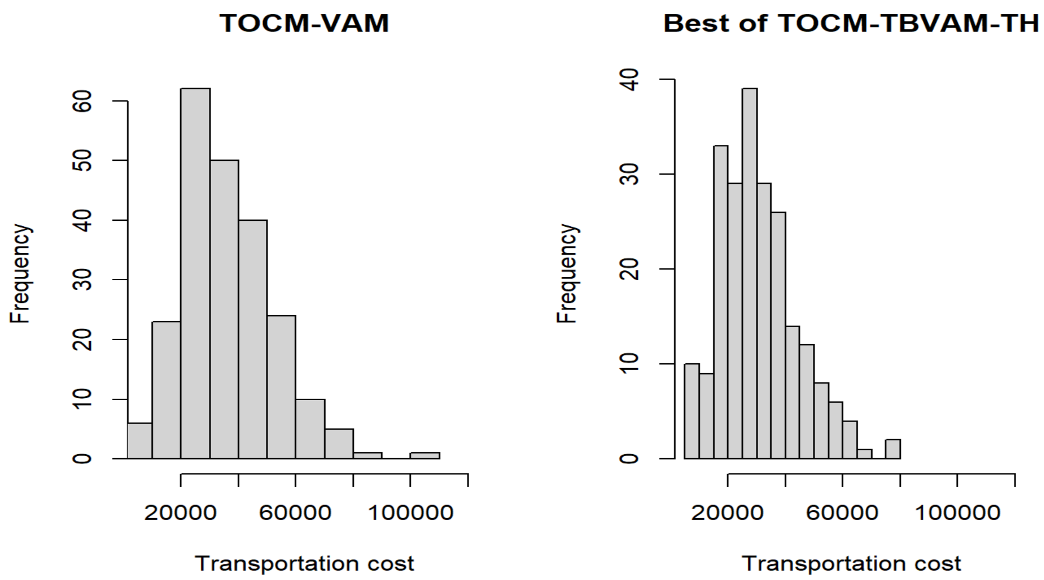

Figure 3: Histogram showing the distribution of the IBFS of TOCM-VAM and best of TOCM-TBVAM-TH.

Figure 3 shows a histogram for one of the sets of 1,000 randomly grouped non-overlapping transportation problems discussed and presented in chapter five. Visual inspection of **Figure 3** indicates that both TOCM-VAM and the best of TOCM-TBVAM-TH are skewed to the right, signifying that the results do not seem to follow a normal distribution. However, as this is only a visual analysis-based suggestion, a more robust statistical test in the form of the Shapiro-Wilk test was conducted on both results to test for normality. The p-value of TOCM-VAM = 1.903e-05 and TOCM-TBVAM-TH = 3.474e-05, both less than 0.05, indicates that there is evidence that suggests both TOCM-VAM and best of TOCM-TBVAM-TH IBFS results do not follow a normal distribution and for that reason, a non-parametric test is required. In addition, the statistical results on the best of TOCM-TBVAM-TH shows a minimum value of 5335.23, a first quartile of 20812.17, a second quartile of 28717.73, third quartile of 38229.66 and a maximum value of 77458.38, with the standard deviation of 13522.51.

To further provide additional findings with the IBFS obtained on TOCM-VAM and TOCM-TBVAM-TH, a two-tailed Wilcoxon signed-rank test, which compares two paired groups, was conducted on the best of TOCM-TBVAM-TH against TOCM-VAM for the full set of 20,000 simulations with the null hypothesis that the distributions of both populations of IBFS costs are identical. We recall that the "best of TOCM-TBVAM-TH" is where there is an improvement over TOCM-VAM when the percentage threshold is applied on the maximum penalty to induce ties in other to gain from the benefit of ties to achieve a lower IBFS with an efficient way of systematically breaking ties at several levels and that both TOCM-VAM and TOCM-TBVAM-TH would produce the same IBFS when there are no ties. The test generated a p-value $< 2.2e-16$ which is less than 0.05, thus rejecting the null hypothesis, which indicates that the distributions of both populations are not identical. Therefore, introducing a systematic method to break ties in the cost optimisation for transportation problems results in an improvement in the IBFS beyond chance effects. Furthermore, the size of the improvement obtained in extensive simulations is 1.4-2.1% with a consistently greater number of wins for TOCM-TBVAM and TOCM-TBVAM-TH compared to the state-of-the-art method, TOCM-VAM, by 2-11%. We also recall that TOCM-TBVAM is the algorithm where no threshold is applied to induce ties, although it also systematically breaks ties at several levels and then shows improvement when the threshold is applied using TOCM-TBVAM-TH on integer cost transportation problems. With the integer cost threshold application, the percentage threshold tends to be much higher to see the effect of improvement on the state-of-the-art method because the cost difference may be greater.

6.2.2 Computational Complexity

Table 27. Computational Complexity for TOCM-VAM and TOCM-TBVAM-TH with various thresholds of 10% interval for 10,000 simulated transportation problems with real-valued costs.

	Time in seconds		
Methods	User	System	Elapsed
TOCM-VAM	**17862.73**	314.02	**18533.87**
TOCM-TBAVM-TH 100	**21901.86**	312.15	**22481.37**
TOCM-TBAVM-TH 90	**21393.94**	320.01	**22129.11**
TOCM-TBAVM-TH 80	**21021.41**	320.28	**21766.61**
TOCM-TBAVM-TH 70	**21113.27**	321.93	**21860.43**
TOCM-TBAVM-TH 60	**21094.17**	319.97	**21850.36**
TOCM-TBAVM-TH 50	**21061.78**	320.86	**21805.96**
TOCM-TBAVM-TH 40	**21035.03**	320.11	**21767.21**

Table 27 shows the computational time of 10,000 simulated transportation problems on a laptop (Intel(R) Processor i3-4030U with 8GB RAM, Windows 10) of both the TOCM-VAM and TOCM-TBVAM-TH algorithms which are presented in seconds. The average number of demand points for these 10,000 simulated transportation problems is 120, while the average number of supply points is 50 which is a relatively large size compared to published articles. Additionally, the size of the transportation

problem is important because fewer supply and demand points would result in less computational time. In **Table 27**, the User time shows the amount of CPU time charged for the user's instruction used to run the code, the System time shows the amount of CPU time charged for the system's execution on behalf of the operating system time, and the Elapsed time is the approximate total of the User and System time. Two of these time executions are significant to this research, first, the User time because we want to know how long it takes for the code to run, and then the Elapsed time to know the length of time it takes the algorithm to complete from beginning to end to produce the IBFS for the 10,000 transportation problems, which is more useful to businesses that require a fast algorithm. Based on the experimental findings, TOCM-VAM's average User time and Elapsed time for a particular transportation problem are 1.79 and 1.85 seconds respectively, while the average User time and Elapsed time for TOCM-TBVAM-TH are 2.12 and 2.20 seconds, respectively. This suggests that TOCM-VAM tends to be faster than TOCM-TBVAM-TH, with an average User time of 0.33 seconds faster and Elapsed time of 0.35 seconds faster. Please note that smaller-size transportation problems may take a shorter time to complete. In general, the User time and Elapsed time of TOCM-VAM indicates that on average it is lower than the average of TOCM-TBVAM-TH, and this is because while TOCM-VAM allocates resources arbitrarily when there are ties, this is done in a shorter amount of time (arbitrarily), while TOCM-TBVAM-TH goes through a systematic analysis to decide on the allocation by first inducing ties and then breaking ties based on a set of predefined rules. As a result, the average computational time of the proposed method will increase. However, since there is an improvement on the state-of-the-art method TOCM-VAM when TOCM-TBVAM-TH is applied, the shortcoming of TOCM-TBVAM-TH is that it would have to run several times to achieve the improvement, and the magnitude of the increase is equal to the number of replications of the method at different proximity thresholds. For example, since there is no precise threshold for the improvement because transportation problems may differ, and when there is no tie in the decision-making process, TOCM-VAM and TOCM-TBVAM-TH would produce the same IBFS, to induce ties, TOCM-TBVAM-TH would have to run multiple times to see the improvement on TOCM-VAM, i.e., TOCM-TBVAM-TH may have to run at 90%, 80%, 70%, 60% and 50% threshold, which is five times as long as TOCM-VAM, please do note that the improvement may be seen at 80% or 70% threshold depending on the transportation problems. The reason for setting a 10% threshold interval is that this steady adjustment may result in a gradual improvement in the quality of the IBFS. This makes this proposed method time-consuming, however, this modest increase in time difference appears insignificant when compared to the better and more efficient IBFS obtained by TOCM-TBVAM-TH over TOCM-VAM.

Additionally, to achieve the time complexity of TOCM-TBVAM-TH, which is the amount of time required for TOCM-TBVAM-TH to run as a function of the length of the input and is a component of computational complexity, we first need to understand that the Big O notation is a standard way to measure the performance of an algorithm on how the time scales with respect to the size of the input. Therefore, we have generated 10 sets of size N by N transportation problems. i.e., equal supply points (rows) and demand points (columns), where N is 10, 20, 50, 100 and 200. i.e., double the scaling of the input. We generated 10 sets of size 10 by 10, 10 sets of 20 by 20, 10 sets of 50 by 50, 10 sets of 100 by 100, and 10 sets of 200 by 200, making a total of 50 simulated transportation problems and recorded the log of average CPU times and the log of the input size N. For the Big O notation, N represents the number of rows plus the number of columns (N="number of demand points" plus "number of supply points"). Hence, we calculate log(N) as N= R + C, where R = rows and C = columns.

Table 28. Average log of CPU times in seconds with log of the size of the input

	Average log CPU times for each algorithm for the values of log(N)		
log(N)	VAM	TOCM-VAM	TOCM-TBVAM-TH
2.995732	-0.63131	-0.5807514	0.1571447
3.688879	0.060346	0.1256629	0.59371
4.60517	1.729048	1.631951	1.928893
5.298317	3.072705	3.014529	3.24227
5.991465	4.553696	4.509223	4.731019

Table 28 shows that on average the proposed algorithm TOCM-TVBAM-TH takes more average log time than VAM and TOCM-VAM as the input size increases. However, to achieve the time complexity, we need to find the slope of the average log of CPU time against the log of scaling size of the input log(N). A visual display of the plot of the average log of CPU time vs the log of scaling in input size log(N) is shown in **Figure 4**.

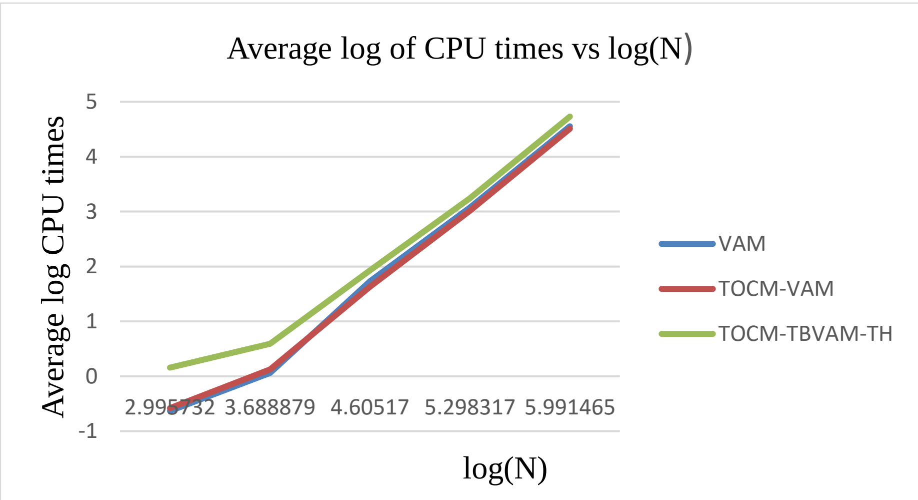

Figure 4: Average log of CPU time for VAM, TOCM-VAM, and TOCM-TBVAM-TH vs log(N).

Using the statistical software tool R, we calculate an estimate of the slope for the three algorithms taking the last two rows, i.e., 100 by 100, and 200 by 200, VAM = 2.13, TOCM-VAM = 2.15 and TOCM-TBVAM-TH = 2.14 all to 2 decimal points. These slopes which are estimates, are similar to the estimates of average scaling derived analytically in Section 3.7, Page 38. They show that the time complexity for VAM, TOCM-VAM and TOCM-TBVAM-TH are similar, with TOCM-TBVAM-TH having the lowest slope estimates, suggesting that inducing ties in the sorting algorithm is efficient. Therefore, the inclusion of the tie-breaking method does not reduce the scaling factor, although it does add a multiplier according to the desired number of thresholds and indeed shifts the corresponding CPU time up as shown in **Figure 4**. We can then conclude that the time complexity of TOCM-TBVAM-TH is $O(N^2 \log(N))$ in the worst-case scenario.

6.3 Discussion

Two methods are proposed in this research, TOCM-TBVAM, and TOCM-TBVAM-TH, while TOCM-TBVAM systematically breaks ties at several levels thereby resulting in an improvement on the IBFS, and TOCM-TBVAM-TH systematically induces ties in real-valued costs transportation problems in the maximum penalty cost by the application of using percentage threshold. This method, TOCM-TBVAM-TH, allows alternative solution paths for the algorithm to find the minimum cost for a given total opportunity cost matrix during the iteration process of allocations. This enables the algorithm to achieve a lower total cost on average than the state-of-the-art, TOCM-VAM. It is also important to note that when ties do not exist, then both TOCM-TBVAM-TH and TOCM-VAM would achieve the same IBFS, however, the benefit of the application of TOCM-TBVAM-TH is an improvement on the IBFS when proximity thresholds in the maximum penalty are applied. As noted in the results obtained, with real-valued cost transportation problems, this algorithm on average outperforms the state-of-the-art algorithm TOCM-VAM, and it is beneficial to users who want to minimise the total cost of product transportation when dealing with logistics and supply chain problems. Also, the use of these proposed algorithms would provide the company with a faster way to reach the optimal solution than the current state-of-the-art algorithm, as on average, the optimisation would start from an improved, higher-quality solution. For example, a company that invests hundreds of millions of pounds in their logistics and supply costs would gain from even a small reduction of 2% in their costs.

Furthermore, an attractive feature of these methods (TOCM-TBVAM and TOCM-TBVAM-TH) is that it is easy to use as it follows the same applications of TOCM-VAM which takes advantage of using the total opportunity cost matrix and then transforms it into a standard transportation problem cost matrix. In addition to that, our proposed methods break ties at several levels more systematically, creating the opportunity to have more minimum costs within an allocation process. Similarly, the concept that TOCM-TVBAM-TH may be used to solve transportation problems and create improvements in IBFS with both real-valued cost and integer cost values is one of its many noteworthy qualities and this can be applied to other existing algorithms that break ties arbitrarily.

6.4 Summary

This chapter presented the findings and general discussion on the research and highlighted that when the total opportunity cost matrix is applied to both the initially proposed algorithm, TBVAM, which transforms to TOCM-TBVAM, and the percentage threshold to induce ties (TOCM-TBVAM-TH), there is an improvement on the state-of-the-art algorithm, TOCM-VAM. The statistical study shows that the results obtained were not due to chance, as the improvement in the IBFS generated by TOCM-TBVAM-TH outperforms that of TOCM-VAM. It should also be recalled that there is no fixed threshold for all transportation problems because the threshold is dependent on the transportation problems, however, a couple of good threshold values to start with seem to be 100% to run without ties, and then 70% on the maximum penalty to induce ties. In addition, the computational complexity of the proposed method is not expensive to run since the improvement in the IBFS obtained compensates for the time required to utilise the proposed method, and of the three algorithms VAM, TOCM-VAM and TOCM-TBVAM-TH, TOCM-TBVAM-TH is more efficient. The following chapter will draw on all the previous chapters to present the conclusion, the novel contribution to knowledge, the limitations of the research, and future research opportunities.

CHAPTER SEVEN

Conclusions and Future Work

7.1 Introduction

The previous chapter discusses the results of the statistical analysis and demonstrates that certainly TOCM-VAM can be improved by utilising a percentage threshold to induce ties at the maximum penalty in other to allow the algorithm several pathways to obtain minimum cost in the iteration process. The chapter also noted that the computational complexity of the proposed algorithm TOCM-TBVAM-TH is slightly higher than the state-of-the-art method, TOCM-VAM but recognised that time spent in the algorithm to induce ties and break these levels of ties in a systematic way does compensate for the improvement on TOCM-VAM and is not expensive to run. This chapter will therefore present the concluding part of the research by linking all previous chapters, detailing the novel contributions to knowledge, the limitation of the research and opportunities for future research.

7.2 Conclusions

The research began with an overview of transportation problems, and the first chapter of this thesis includes the research motivations, research questions, the research aims and objectives, and finally the expected contribution to knowledge of the research. Chapter two of the thesis presented a broad literature review on transportation problems, with a focus on some of the published literature on how to obtain the IBFS for transportation problems, as well as their shortcomings and advantages. It was this literature that affirmed that TOCM-VAM was the state-of-the-art method for finding the IBFS for transportation problems since it is an efficient method for providing an IBFS that is close to or same as the optimal solution. However, it does have the shortcoming of making arbitrary allocations in the decision process when it encounters a tie in the maximum penalty cost, minimum cost and quantity of the demand that can be satisfied. This observation highlighted the research gaps which then led to part of the research motivation.

The lack of publications incorporating real-valued costs in the cost matrix and publications where methods are tested on transportation problems with significant numbers of demand and supply points were noted as research gaps.

In chapter three of this thesis, the two distinct types of transportation problems, balanced and unbalanced, were described together with their mathematical formulations. Chapter three also presented three commonly used methods for finding the IBFS for transportation problems, namely the Northwest Corner Method, the Least Cost Method, and the Vogel Approximation Method. It was found that due to the many iterations in the decision-making process involved, the Vogel Approximation Method does require a bit more computational time than the other two algorithms to produce an IBFS; however, this computational time is offset by the high quality of the IBFS that is produced. The two methods of evaluating optimality, MODI, and Stepping-Stone, were also covered in this chapter. It was noted that before using either of these approaches, it is vital to obtain an IBFS because it will take less time to achieve the optimal situation. The proposed methods, TBVAM, TOCM-TBVAM, and TOCM-TBVAM-TH, were introduced in Chapter Four. TOCM-VAM, which is the state-of-the-art method was a better comparison than the initially proposed algorithm, TBVAM, therefore TOCM-TBVAM and TOVCM-TBVAM-TH were developed. The benefit of these proposed methods is that they all break ties at various levels in the iteration process; however, TOCM-TBVAM and TOCM-TBVAM-TH only break ties once they have obtained the total opportunity cost matrix and then transform to the cost matrix of transportation problems which is then used for the decision-making process. The unique feature of TOCM-TBVAM-TH is that it not only breaks ties at several levels but also induces ties at the first level of tie-breaking which is at the maximum penalty, giving the algorithm multiple pathways for decision-making by allowing a range of costs to be considered as minimum costs to achieve a lower IBFS. The procedures for computing TBVAM, TOCM-TBVAM, and TOCM-TBVAM-TH were also provided, with TOCM-TBVAM-TH just being an extension of TOCM-TBVAM because of the application of threshold proximity to the maximum penalty cost. Two examples of transportation problems from the literature were taken into consideration, and it was noted that TOCM-VAM and TOCM-TBVAM would both produce the same IBFS in the absence of ties, but that TOCM-TBVAM would, on average, produce a lower IBFS in the presence of ties. The 35 benchmark balanced transportation problem results were also shown, including the number of wins and the average proportional difference between the best TOCM-TBVAM-TH and TOCM-VAM, which showed an improvement over TOCM-VAM. Chapter Five presented how the real-valued costs in 20,000 simulated transportation problems were generated to test the efficiency of TOCM-TBVAM-TH. It also noted that the reason for these generated datasets was to reflect real-world cases of transportation problems with large size dimensions which was one of the identified gaps in the literature. The results and analysis from the 20,000 simulated transportation problems were reported in Chapter Six, and it was noted that the result achieved by TOCM-TBVAM-TH, which was an improvement over TOCM-VAM, was not the result of chance. Although it was noted that TOCM-TBVAM-TH takes more time to induce and then continuously break ties at various levels in the decision-making process, the additional computational complexity of TOCM-TBVAM-TH is offset by the improvement in the resulting IBFS, which is lower and reduces the amount of time needed to achieve the optimal solution. Another suggestion is to compute the TOCM-TBVAM-TH twice at two different percentage thresholds, such as 100% and then 70%. 70% because it is just an indicative threshold based on the simulated transportation problems tested and there is no set threshold to be taken because all transportation problems may differ in the cost matrix, and demand and supply size dimensions, and 100% because it also represents TOCM-VAM where no ties are considered.

7.3 Novel Contribution to Knowledge

This research has been able to adequately provide answers to the research questions outlined in the first chapter of the thesis. First, the research was able to provide both TOCM-TBVAM and TOCM-TBVAM-TH to address the limitations of TOCM-VAM which is making arbitrary allocations in cases of ties. This study showed that the proposed methods can be used to improve the IBFS for transportation problems where ties exist at some levels of the decision-making process as seen in the results obtained in the research presented in **Table 23**, **Table 24,** and **Table 26**. Second, the benefit of the proposed algorithms of inducing ties and breaking ties does improve the IBFS obtained when compared to the state-of-the-art method, TOCM-VAM. Such benefits include achieving a significantly lower IBFS, which would take less time to achieve the optimal solution and be beneficial for businesses looking to cut costs in their supply chain and logistics. This proposed algorithm would undoubtedly be a top option to consider when solving transportation problems. Lastly, the application of threshold proximity to induce ties in real-valued cost has also shown to be an improvement over the state-of-the-art method with the results obtained in **Table 26**.

Therefore, the research contributions to knowledge are as follows:

- The proposed algorithms TOCM-TBVAM and TOCM-TBVAM-TH can improve on TOCM-VAM's IBFS by breaking ties at several levels such as maximum penalty cost, minimum cost, and quantity supplied which tends to be the limitation of the state-of-the-art algorithm TOCM-VAM.
- TOCM-TBVAM-TH for real-valued cost transportation problems can improve TOCM-VAM's IBFS by considering a certain percentage range to be taken as indicative of ties to give the algorithm different pathways to consider minimum cost in the maximum penalty.
- My simulations shows that it is feasible to achieve improvements in the number of wins on average by about 2% when we consider a range of percentage thresholds with minimum additional effort. This level of improvement can have a significant impact on business costs.

7.4 Research Limitations

The main limitation of this research is that the proposed methods were tested only for balanced transportation problems. Although unbalanced transportation problems can be converted to a balanced transportation problem by adding dummy demand and supply points appropriately, this is beyond the scope of this thesis.

7.5 Future Research

There are opportunities for future research due to the time limit for completing the PhD. Therefore, the following are the future research opportunities:

- With this use of threshold proximity on maximum penalty cost, there is a need for more research. Any algorithm that resolves ties in the maximum penalty cost, for example, may benefit from having a range of minimum costs within that iteration, resulting in a lower IBFS than when the maximum penalty cost is at an exact value.

- Another potential area for future research is to look two steps ahead in the iteration and see how it can implement the best move. The limit of only looking two steps ahead is in place to keep the number of possibilities manageable.
- Furthermore, the opportunity for future research can be discussed on how to break ties when there is a tie in the maximum demand quantity that can be satisfied as these proposed methods make arbitrary allocations in cases when there is a tie in the maximum demand quantity that can be satisfied.

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Appendix 1

R Code for the Simulated Transportation Problems

```
## R Code for Simulated Transportation Problems with Continuous Cost
#Clears the memory
rm(list=ls(all=TRUE))

library(openxlsx)
### nprobs denotes the number of transportation problems that need to be simulated
nProbs = 2000
### rangeDemand denotes the minimum and maximum demand quantities.
rangeDemand = c(100, 1000)
#### rangeSupply denotes the minimum and maximum supply quantities.
rangeSupply = c(500, 5000)
###### nDem denotes the minimum and maximum demands points
nDem = c(50, 200)
### nSup denotes the minimum and maximum supply points
nSup = c(10, 100)
### rangeCost denotes the minimum and maximum cost range for the simulated TP
rangeCost = c(50, 150)
##### costChange controls the variation (min & max) values of random costs around
a randomly drawn central mean value
costChange = 10

for (i in 1:nProbs)
{
  print(sprintf("Creating file %d of %d ...", i, nProbs))
  n1 = round(runif(1, min = nDem[1], max = nDem[2]), digits = 0)
  demand = round(runif(n1, min = rangeDemand[1], max = rangeDemand[2]), digits
= 0)

  n2 = round(runif(1, min = nSup[1], max = nSup[2]), digits = 0)
  supply = round(runif(n2, min = rangeSupply[1], max = rangeSupply[2]), digits = 0)
  ### ensures transportation problem is balanced
  supply = round((supply / sum(supply)) * sum(demand), digits = 0)

  pick = 1:n2

  while(sum(supply) < sum(demand))
  {
    k = sample(pick, 1, replace = TRUE)
    supply[k] = supply[k] + 1
  }


  avg_cost = round(runif(1, min = rangeCost[1], max = rangeCost[2]), digits = 0)

  centerCostLower = round(avg_cost - ((avg_cost * (costChange / 100))) / 2, digits =
0)
```

```
  centerCostUpper = round(avg_cost + ((avg_cost * (costChange / 100))) / 2, digits =
0)

  costs = matrix(round(
    runif(n1 * n2, min = centerCostLower, max = centerCostUpper),
    digits = 2
  ),
  nrow = n2, ncol = n1)

  prob = rbind(costs, demand)
  prob = cbind(prob, c(supply, sum(supply)))

  #fname = "place to save files"

  fname = "C://Users//Ltopp//Desktop//sample dataset"
  fname = sprintf("%s/tb%d_%d_%d.xlsx", fname, i, n1, n2)

  prob = as.data.frame(prob)
  colnames(prob) = c(sprintf("D%d", 1:(ncol(prob)-1)), "Supply")
  rownames(prob) = c(sprintf("S%d", 1:(nrow(prob)-1)), "Demand")

  write.xlsx(prob, file = fname,
             colNames = TRUE,
             rowNames = TRUE)
}

print(prob)
```

Appendix 2

Website for the datasets of Transportation Problems

The Simulated transportation problems can be seen and downloaded on the following website.

1) 35 Benchmark Dataset

   https://www.mediafire.com/file/efepl8bl286t2ev/35.zip/file

2) First 10,000 Simulated Dataset with real-valued cost

   https://www.mediafire.com/file/3rqlb00qudaxzpw/10000_simulated_TP_with_real_valued_cost.zip/file

3) Second 10,000 Simulated Dataset with real-valued cost

   https://www.mediafire.com/file/4alg9gqdcom8snm/10000_simulated_TP_with_Real_valued_cost_2.zip/file