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Abstract

Metaheuristics play a major role in the important domain of global optimization. Since they are problem independent, they can be effectively used in constrained and higher-dimension problems and in real-world applications also. In this paper, one novel two-phase trigonometric algorithm is presented to obtain optimal/ near-optimal solutions for different optimization problems. In this work, the main focus is given to solving real-world engineering problems with constraints. The proposed algorithm effectively explores and exploits the search space to arrive at the solutions. Benchmarks analyzed include unconstrained (unimodal and multimodal) functions, constrained special functions, constrained engineering problems and ten problems of the "100 digit challenge", Institute of Electrical and Electronics Engineers (IEEE) Congress on Evolutionary Computation (CEC2019) totaling fifty-nine problem instances. The problems include functions with continuous variables, discrete variables and both continuous and discrete variables. They are all single-objective functions that are suitably modelled and simulated in MATLAB environment. The obtained results are compared with recent and time-tested popular algorithms including Differential Evolution (DE), Improved Teaching–Learning-Based Optimization (ITLBO), Social Network Search (SNS), Firefly Algorithm (FA), Cuckoo Search (CS) and Whale Optimization Algorithm (WOA). Analyses of the results obtained from TP-AB against the performance of the well-known algorithms indicate the superiority and competitiveness of the proposed algorithm in providing quality solutions.

Keywords Optimization · Constrained and un-constrained optimization · Benchmark · Engineering problems · Population-based algorithm

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1 Introduction

Optimization refers to identifying the best feasible solution for a problem in a given search space. Optimization invariably finds its place in almost all the domains of industry, economics, industry and science (Dehghani et al., 2022). Many mathematical tools are used to obtain exact solutions for smaller problems. However, the computation time and cost increase exponentially as the size of the problem grows. As a result, it may not be possible to obtain the exact solution using traditional tools. Several heuristics and problem-independent metaheuristics are proposed by researchers over the past five decades that are capable of yielding approximate solutions with an acceptable level of errors (Sarhani et al., 2023). Due to the amount of computing power available today, solving optimization problems with higher dimensions and more constraints has become a reality and more popular (Long et al., 2019). Any optimization method may be deterministic or stochastic (Wang et al., 2019). Deterministic methods generally use gradient-based information for finding the global optimal solution. Stochastic methods generate and use random variables within the bounds and move towards the optimal/ near-optimal solution. Population-based algorithms fall under this category and are prevalent nowadays. These algorithms have similarities to social, natural and physical systems that can be effectively modelled and used as optimizers for solving problems. Initial approximate solutions are randomly generated and moved towards the best feasible solution using the intelligence of the algorithms through several iterations (generations).

Since the introduction of the Genetic Algorithm (Holland, 1992), different versions of evolutionary algorithms have been proposed by several authors over the past decades. Arithmetic Operators (Abualigah et al., 2021) and Trigonometric functions are also used in many optimization algorithms (Baskar, 2022a, 2022b; Mirjalili, 2016) for solving unconstrained and constrained optimization problems. Grey Wolf Optimizer (GWO) of Mirjalili et al., 2014 which mimics the leadership hierarchy and hunting mechanism of grey wolves is one of the nature-inspired better performing algorithms of recent years. The Teaching Learning Based Optimization (TLBO) (Rao et al., 2011) that works on the effect of the influence of a teacher on learners is also another population-based nature-inspired algorithm. Even the social networking feature is modelled to arrive at an efficient "Social Network Search (SNS)" algorithm (Bayzidi et al., 2021) which is proved to be one of the most efficient algorithms of recent times. Some of the better-performing and novel algorithms are presented in Table 1.

Any algorithm may be devised exclusively for solving a specific problem or can be a more generalized one. The popular "no-free-lunch theorem of optimization (Wolpert & Macready, 1997)" concludes that it is impossible to have a general-purpose, universal optimization strategy for all categories of problems. The only way one particular approach can outperform another one is if it is focused on the core structure of a specific problem under discussion (Ho & Pepyne, 2002). This implies that there is clear scope for a new algorithm in the broad domain of "optimization" Motivated by this, a new two-phase trigonometric algorithm TP-AB ("TP" represents "Two-Phase" and "AB" represents the first two letters of the author who devised the concept) is proposed for solving both constrained and unconstrained optimization problems. A total of fifty-nine single objective, constrained and unconstrained benchmark functions and engineering problems are solved and analyzed. Finally, the test suite of CEC2019 is analyzed and compared with a few other algorithms. The proposed new algorithm can also be used with a tuning parameter if required. In this work, a tuning parameter of $\boldsymbol{\alpha} = \mathbf{1}$ is considered in all the problems except one in which $\boldsymbol{\alpha} = \mathbf{2}$. The better exploring and exploiting powers are proved by the results when compared with several better-performing algorithms available in the literature.

Table 1 Few popular metaheuristic algorithms

S. no	Algorithm	Author(s)
1	Arithmetic optimization Algorithm (AOA)	Abualigah et al. (2021)
2	Atomic Orbital Search (AOS)	Azizi et al. (2021)
3	Chaos Game Optimization (CGO)	Talatahari and Azizi (2020)
4	Cuckoo Search (CS)	Yang and Deb (2010)
5	Differential evolution (DE)	Storn and Price ((1997))
6	Firefly Algorithm (FA)	Gandomi et al. (2011)
7	Genetic Algorithm (GA)	Holland (1992)
8	Gravitational Search Algorithm (GSA)	Rashedi et al. (2009)
9	Grey wolf optimizer (GWO)	Mirjalili et al. (2014)
10	Multilevel Cross Entropy Optimizer (CEO)	MiarNaeimi et al. (2018)
11	Particle Swarm Optimization (PSO)	Kennedy and Eberhart (1995)
12	Sine Cosine Algorithm (SCA)	Mirjalili (2016)
13	Social Network Search (SNS)	Bayzidi et al. (2021)
14	Teaching Learning Based Optimization (TLBO)	Rao et al. (2011)
15	Water Cycle Algorithm (WCA)	Eskandar et al. (2012)
16	Water Strider Algorithm (WSA)	Kaveh AND Eslamlou (2020)
17	Whale Optimization Algorithm (WOA)	Mirjalili and Lewis (2016)

The rest of the paper is organized as follows. The proposed algorithm is detailed in Sect. 2 along with pseudo code and a flowchart. The in-depth analysis performed in this work along with the comparisons made are presented in Sect. 3, mainly based on recent bibliographic references. In Sect. 4, some benchmarks and engineering problems are presented followed by detailed analysis of the results and comparison with few other popular algorithms in Sect. 5. Finally, in Sect. 6, the it is concluded with a summary and the advantages of the proposed algorithm.

2 Proposed algorithm

Better "exploitation" and "exploration" are two important features required for any optimization algorithm. "Exploration" ensures visiting entirely new regions of the search space during the iteration process whereas; "exploitation" searches the neighborhoods of the already visited points. Hence, the probability of better convergence increases if both are well taken care of. If exploitation and exploration are enforced in separate phases (Rao et al., 2011), the chances of better search improves.

Adding or subtracting a fraction of an approximate solution is a way of exploiting the neighbourhoods of that solution, that is, $\boldsymbol{X}$ $\boldsymbol{f*X}$, "$\boldsymbol{f}$± is a fraction between [0 1]. There are three easy ways of achieving " $\pm f$"; **(*i*) −1 + 2**rand*(*ii*)*Sin*(2**pi***rand*)*and*,(*iii*)*Cos*(2**pi***rand*)**, '***rand***' is a positive number between [0 1]. The first method generates numbers from−1 to + 1 uniformly whereas, the other ways generate differently. That is, the new solution generated significantly differs in each case even for the same random number value resulting in different exploitations.

Case (i) $-1 + 2*rand$: 1 to 0 to 1 uniformly for random numbers from 0 to 1.

Case (ii) $Sin(2*pi*rand)$: 0 to 1(for random values 0 to 0.25), 1 to 0 (for random values 0.25 to 0.50), 0 to −1 (for random values 0.50 to 0.75) and, −1 to 0 (for random values 0.75 to 1.0).

Case (iii) $Cosin(2*pi*rand)$: 1 to 0 (for random values 0 to 0.25), 0 to −1 (for random values 0.25 to 0.50), − 1 to 0 (for random values 0.50 to 0.75) and, 0 to 1 (for random values 0.75 to 1.0).

The result is that each strategy follows a different pattern of exploitation.

To improve the exploration effectiveness, the second phase adopts a slightly different logic; the '*i*th" solution in the population is updated by,

$X_i = X_i \pm f*[X_i - X_j]$; where '$X_j$" is the neighbour in front of "X_i". Since $[X_i - X_j]$ is highly stochastic, this second phase is mainly accountable for the exploration of a new region in the search space. Motivated by these features, a new algorithm is devised.

The proposed two-phase algorithm initially generates a pool of approximate solutions. The first phase exploits the neighbourhoods of the best solution obtained during the earlier iteration. The second phase explores a new region within the search space. Thus, combining both phases in the same iteration improves the solution quality rapidly towards the optimal/ near-optimal solution. The expressions used are:

$$\text{Initial Population}: X = LB + rand * (UB - LB) \tag{1}$$

$$\text{First Phase}: X_i = X_i + Sine(2 * pi * rand) * X_i \,(\text{without tuning parameter}) \tag{2}$$

$$\text{Second Phase}: X_i = X_i + Sine(2 * pi * rand) * \left(X_i - X_j \ldots j = i + 1\right)\left(\text{without tuning parameter}\right). \tag{3}$$

where LB—Lower Bound; UB—Upper Bound; $rand$—a random number between [0 1]; X—Solution set and, X_i—Solution set during '*i*th' iteration.

"Tuning Parameter" narrows down the search space when the iteration progresses.

Cosine function also could be used in both phases instead of Sine.

The "Penalty" method is used for transforming a constrained problem into an unconstrained one (Kuri-Morales & Gutiérrez-García, 2002; Yeniay, 2005). The additive form is applied in this paper which is expressed as follows:

$$Evaluate(\bar{x}) = \begin{cases} f(\bar{x}), & if\ (\bar{x}) \in F \\ f(\bar{x}) + p(\bar{x}), & otherwise \end{cases} \tag{4}$$

where $p(\bar{x})$ represents a penalty term. If no violation of any constraint occurs, $p(\bar{x})$ will be zero and a positive value otherwise. The strategy used is:

$$p(x) = \sum_{j=1}^{c} (10^9 * g(j)) \tag{5}$$

where ***c***—number of constraints, ***g(j)***—violation in constraint ‘***j***’ if any.

That is, $p(\bar{x})$ **0** *if* $g(j)$ **0** *else*, $p(x)$ $p(\bar{x})$.

A penalty value of 10^9 is considered for all the problems except the “Stepped Cantilever Beam (Discrete)” for which the value is 10^3. All the problems solved in this work have only inequality constraints.

Hence, if an optimum value is reported, it implies that all the constraint values are less than zero. The pseudo-code is presented in Fig. 1.

```
Randomly initialize the solutions: X                                   % N Solutions
Compute the Costs
Select the "Best Cost" and "Best Solution"
While (Current_Iteration < Maximum_Iteration) do                      % while loop starts
    New Set of Solutions: NewX = X + Sin (2*pi*rand) * X               % N solutions
    Trim
    Compute the Costs %%%%% FE-I
    Concatenate the Solution Sets and Costs: [X NewX]                  % 2*N solutions
    Sort the Costs in ascending order
    Select the "Best Cost" and "Best Solution"
    Take the best half of the solutions; discard the other half: X     % N Solutions
      for (i=1 to N) do                                               % for loop starts
        if i==N, j=1
        else j=i+1
        end if
        Improved Solution: X1 = X(i) + Sin (2*pi*rand)*(X(i) - X(j))
        Trim
        Compute the cost of improved solution: Cost(X1) %%%%% FE-II
        if Cost(X1) < Cost(X(i))
          X(i)=X1; Cost(X(i))=Cost(X1)
        end if
        if Cost(X1) < "Best Cost"
          "Best Solution" = X1; "Best Cost" = Cost(X1)
        end if
      end for                                                         % for loop ends
Current_Iteration = Current_Iteration+1
end while                                                             % while loop ends
Return the "Best Cost"; "Best Solution"
```

Fig. 1 Pseudo-code of the algorithm

When a tuning parameter "***a***" is used, the expressions are modified in line numbers 5 and 16 of the pseudo-code as:

$$NewX = X + r * Sin(2 * pi * rand) * X \tag{6}$$

$$X1 = X(i) + r * Sin(2 * pi * rand) * (X(i) - X(j)) \tag{7}$$

$$r = a - (a * t/T) \tag{8}$$

t current iteration number and, ***T*** maximum number of iterations.

At the beginning of the iteration, "***r***" will be equal to the tuning parameter "***a***" and zero during the last iteration. In this work the used defined parameter, "***a*** **1**" unless otherwise specified.

From the pseudo-code, it is clear that the algorithm evaluates the cost function twice during each iteration (lines 7 and 18). Hence, the number of iterations should be halved when the results are to be compared with other algorithms. Throughout this paper, the actual population size and iterations used are specified which has to be multiplied by 2 if the number of function evaluations (FEs) is required for comparison. In the first phase of the algorithm (serial numbers 8 to 11 of the pseudo-code), "greedy selection" also could be employed instead of the (μ + λ) strategy in line with the second phase (serial numbers 19 to 25 of the pseudo-code).

The algorithm is represented in the form of a flowchart also and presented in Fig. 2.

The complexity of the new algorithm is:

Initialization: O(N); N—population size.

Updating: O(N) + O(2*I*N*D); I—number of iterations, D—dimensions (variables) and 2 sets of function evaluations per iteration.

Complexity: O(N*(2*I*D + 1)).

The objective of this work is to propose a novel algorithm that can be used effectively for solving unconstrained, constrained and constrained real-world engineering problems. A wide variety of benchmark problem instances are analyzed as detailed below:

- Ten Unconstrained Benchmark Problems: dimensions 30, 50 and 100
- Seven Unimodal Unconstrained Benchmark Functions: dimensions 30, 100, 500 and 1000
- Six Multimodal Unconstrained Benchmark Functions: dimensions 30, 100 and 1000
- Ten Constrained Special Functions: dimensions two to ten and number of constraints one to eight
- Sixteen Constrained Engineering Problems: dimensions two to eleven and constraints zero to eleven
- Ten problems of CEC2019, "100 digit challenge".

The proposed algorithm is compared with several better-performing algorithms available in the literature. The results demonstrate the better performance of TP-AB over many of the compared popular algorithms in solving a wide range of optimization problems.

3 Unconstrained optimization problems

Initially, ten unconstrained problems are considered the optimal value of all is zero (Table 2). The first five are scalable functions and the remaining has fixed dimensions. A population size (PS) of 10 is taken with a maximum of 2000 iterations (IT) for 30 dimensions and 2500 iterations for dimensions 50 and 100. 30 trials are conducted and the 'Minimum', "Mean", "Maximum" and "Standard Deviation (SD)" are computed. TP-AB without tuning is considered for this set of functions.

Similar analyses were carried out by Yu et al. (2016) and; Differential Evolution (DE, Storn & Price, 1997), Artificial Bee Colony (ABC) algorithm (Karaboga, 2005) Gbest-guided Artificial Bee Colony algorithm (GABC, Zhu & Kwong, 2010) and Improved Teaching–Learning based Optimization (ITLBO, Yu et al., 2016) algorithms were used for comparison purpose. Since the number of function evaluations is the same, the results are compared with the present work.

The "*Mean*" values and standard deviation are reproduced and presented in Table 2 itself.

The analysis shows that TP-AB performs well for the "*Mean*" values in all the problems and dimensions except, Rosenbrock and Schwefel2.21 functions in which the ABC algorithm and ITLBO are the best performers. For Sphere, Griewank and Schwefel2.22 problems; both ITLBO and TP-AB report optimal values. In the case of Ackley, Rastrigin, Step, Schwefe1.2 and Quartic benchmarks, TP-AB is a clear winner.

Now, the population size is reduced to 5 from 10 and the number of iterations is raised to 4000 to keep the FEs the same and the outputs are taken. The results are presented in Table 3 for 30 dimensions for both 5 and 10 population sizes.

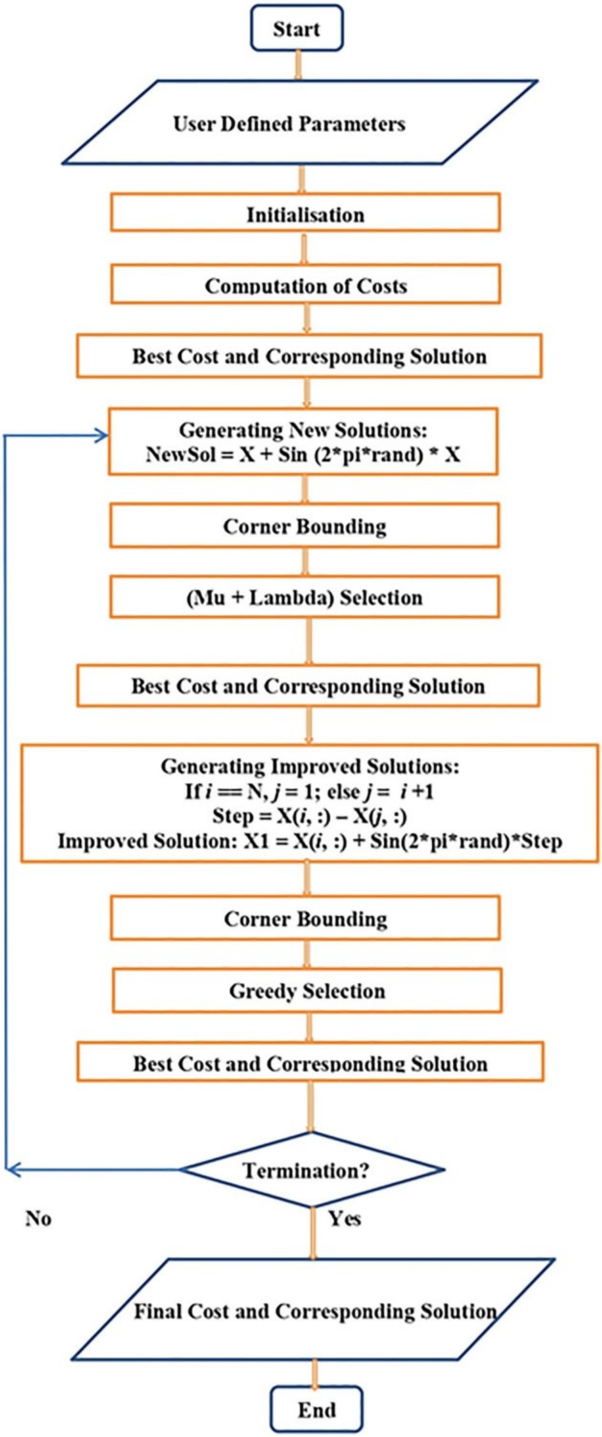


Fig. 2 Flowchart of the proposed algorithm

Table 2 Comparison of results for unconstrained benchmark problems

Function	Dim	Mean	Algorithm	SD	Algorithm	TP-AB (mean)	TP-AB (SD)
Sphere	30	0	ITLBO	0	ITLBO	0	0
	50	0	ITLBO	0	ITLBO	0	0
	100	0	ITLBO	0	ITLBO	0	0
Rosenbrock	30	**4.55**	**ABC**	0.146	ITLBO	25.8016	0.2920
	50	**4.33**	**ABC**	0.556	ITLBO	46.5577	0.7301
	100	**90.9**	**ITLBO**	0.646	ITLBO	97.6741	0.6310
Ackley	30	3.55E−15	ITLBO	0	ITLBO	**8.88E−16**	0
	50	3.55E−15	ITLBO	0	ITLBO	**8.88E−16**	0
	100	3.78E−15	ITLBO	9.01E−16	ITLBO	**8.88E−16**	0
Griewank	30	0	ITLBO	0	ITLBO	0	0
	50	0	ITLBO	0	ITLBO	0	0
	100	0	ITLBO	0	ITLBO	0	0
Rastrigin	30	3.31E−02	GABC	0.181	GABC	**0**	0
	50	0.472	ABC	0.492	ABC	**0**	0
	100	0	ITLBO	0	ITLBO	0	0
Step	30	1.72E−27	DE	1.39E−27	DE	**0**	0
Schwefel2.22	30	0	ITLBO	0	ITLBO	0	0
Schwefe1.2	30	4.38E−322	ITLBO	0	ITLBO	**0**	0
Schwefe2.21	30	**0**	ITLBO	0	ITLBO	4.7628e−238	0
Quartic	30	2.05E−04	ITLBO	1.96E−04	ITLBO	**1.3726e−005**	1.3300e-005

Table 3 Results of 30 dimensions for different population sizes (FEs remain the same)

Function (PSxIT)	Min	Mean	Max	SD	Mean FEs
Sphere					
10 × 2000	0	0	0	0	9.7673e+003
5 × 4000	0	0	0	0	5387.5
Rosenbrock					
10 × 2000	25.4426477	25.8015620	26.6124803	0.2919683	20,000
5 × 4000	24.818	25.562	26.099	0.2763	20,000
Ackley					
10 × 2000	8.88E−16	8.88E−16	8.88E−16	0.00E + 00	20,000
5 × 4000	4.4409e−16	4.4409e−16	4.4409e−16	0	20,000
Griewank					
10 × 2000	0	0	0	0	732
5 × 4000	0	0	0	0	428.33
Rastrigin					
10 × 2000	0	0	0	0	581.3333
5 × 4000	0	0	0	0	310.17
Step					
10 × 2000	0	0	0	0	223
5 × 4000	0	0	0	0	137
Schwefel2.22					
10 × 2000	0	0	0	0	1.5345e+004
5 × 4000	0	0	0	0	8275.7
Schwefe1.2					
10 × 2000	0	0	0	0	1.5447e+003
5 × 4000	0	0	0	0	861
Schwefe2.21					
10 × 2000	7.1658e−272	4.7628e−238	9.0031e−237	0	20,000
5 × 4000	0	0	0	0	1.6050e+04
Quartic					
10 × 2000	5.2588e−008	1.3726e−005	4.8087e−005	1.3300e−005	20,000
5 × 4000	4.0171e−07	9.3731e−05	2.0228e−04	5.9022e−05	20,000

The results show that the performance improves (min, mean, max, SD and average FEs) except for the "Quartic Function" when the population size is reduced to 5 from 10 and keeping the number of function evaluations the same by increasing the number of iterations.

3.1 Thirteen more unconstrained problems

To further assess the performance of the new algorithm, thirteen more problems used by the Arithmetic Optimization Algorithm (AOA, Abualigah et al., 2021) are considered (Tables 4 and 5). The authors used 30,000 function evaluations for different algorithms (PS: 30, IT: 1000). In this paper, the function evaluations (FEs) are maintained as the same with a different

Table 4 Unimodal benchmark functions

Function	Expression, f(x)	Dimension	Search range	Optimal value
F1	$\Sigma x_i^{\,2}, i = 1 to\ n$	30, 100, 500, 1000	[−100, 100]	0
F2	$\Sigma\|x_i\| + \Pi\|x_i\|, i = 0 to\ n$	30, 100, 500, 1000	[−10, 10]	0
F3	$\Sigma_{i=1}(\Sigma_{j=1}^{i} x_j)^2$	30, 100, 500, 1000	[−100, 100]	0
F4	$Max_i \{\|x_i\|, 1 \leq i \leq n\}$	30, 100, 500, 1000	[−100, 100]	0
F5	$\Sigma[100*(x_i^{\,2} - x_{i+1})^2 + (1 - x_i)^2], i = 1$ to $(n - 1)$	30, 100, 500, 1000	[−30, 30]	0
F6	$\Sigma([x_i + 0.5])^2, i = 1 to\ n$	30, 100, 500, 1000	[−100, 100]	0
F7	$(\Sigma i*x_i^{\,4}) + random[0, 1], i =$ 0 to n	30, 100, 500, 1000	[−128, 128]	0

Table 5 Multimodal benchmark functions

Function	Expression, f(x)	Dimensions	Search range	Optimal value
F8	$\Sigma(-x_i*Sine(\sqrt{\|x_i\|})), i = 1 to\ n$	30, 100, 1000	[−500, 500]	- 418.9829*n
F9	$\Sigma[x_i^{\,2} - 10*Cosine(2*\pi*x_i) + 10]$	30, 100, 1000	[−5.12, 5.12]	0
F10	$-20*exp(-0.2*\sqrt{\frac{1}{n}*\sum_{i=1}^{n} x_i^2}) - exp(\frac{1}{n}*\sum_{i=1}^{n} Cosine(2*\pi*x_i)) + 20 + e$	30, 100, 1000	[−32, 32]	0
F11	$1 + (1/4000)*\sum_{i=1}^{n} x_i^2) - \Pi[cosine(x_i/\sqrt{i})], i = 1 to\ n$	30, 100, 1000	[−600, 600]	0
F12	$(\pi/n)*[10*Sine(\pi*y_i)] + \sum_{i}^{n-1}(y_i - 1)^2 [1 + 10*Sine^2(\pi*y_i + 1) + \sum_{i=1}^{n} u(x_i, 10, 100, 4)$	30, 100, 1000	[−50, 50]	0
F13	$0.1*(sine^2(3*\pi*x_i) + \Sigma(x_i - 1)^2*[1 + Sine^2(3*\pi*x_i + 1)] + (x_n - 1)^2 + Sine^2(2*\pi*x_n)) + \Sigma u(x_i, 5, 100, 4), i = 1 to\ n$	30, 100, 1000	[−50, 50]	0

population size of 10. Uniformly 30 trials are carried out and “Min.”, “Mean”, “Max.” and “SD” values are computed.

The results are referred from the AOA paper and compared with the results of TP-AB (Table 6) for dimensions 30, 100, 500 and 1000.

The results are extracted from the AOA paper (Abualigah et al., 2021) and the algorithms are ranked individually for each problem and dimension (Table 7). Table 8 shows the summary of ranking for dimensions 30, 100, 500 and 1000. AOA is ranked one when all the dimensions are considered followed by TP-AB and, GWO comes next. TP-AB performs better in 100

Table 6 Results of functions: F1 to F13 [TP-AB, PS: 10; It: 1500, Trials: 30]

Function	Dimensions	Min	Mean	Max	STD	Mean FEs
F1	30	0	0	0	0	9.7673e+003
	100	0	0	0	0	1.2125e+004
	500	0	0	0	0	13,601
	1000	0	0	0	0	1.4322e+04
F2	30	0	7.1575e−309	2.0864e−307	0	1.4995e+004
	100	3.7428e−282	2.0937e−266	3.6913e−265	0	15,000
	500	1.1429e−257	1.2970e−241	3.0738e−240	0	15,000
	1000	1.7677e−249	1.5888e−235	2.9152e−234	0	15,000
F3	30	0	0	0	0	1.0942e+004
	100	0	0	0	0	1.3813e+004
	500	1.4446e−319	2.1976e−244	6.5928e−243	0	15,000
	1000	2.1662e−300	2.9385e−227	8.8156e−226	0	15,000
F4	30	7.3622e−203	3.0302e−156	6.1497e−155	1.2277e−155	15,000
	100	94.5495	97.2759	98.9782	1.0658	15,000
	500	98.8218	99.4585	99.8636	0.2491	15,000
	1000	99.4768	99.7093	99.8845	0.1101	15,000
F5	30	25.6367	26.1388	27.0351	0.3383	15,000
	100	96.2587	97.9800	98.5157	0.5848	15,000
	500	497.7702	498.4099	498.5329	0.1486	15,000
	1000	998.2538	3.0381e+009	1.5860e+010	6.1828e+009	15,000
F6	30	1.0835e−004	0.0017	0.0134	0.0029	15,000
	100	9.6877	11.4012	12.7533	0.8647	15,000

Table 6 (continued)

Function	Dimensions	Min	Mean	Max	STD	Mean FEs
	500	108.33	110.65	114.29	1.2410	15,000
	1000	232.6969	235.8135	238.0111	1.3150	15,000
F7	30	3.3939e−07	2.3858e−05	7.3183e−05	2.0240e−05	15,000
	100	3.9205e−007	2.2784e−005	7.2759e−005	2.0695e−005	15,000
	500	5.6305e−007	2.5029e−005	9.9904e−005	2.5795e−005	15,000
	1000	6.1854e−007	4.1779e+012	2.5740e+013	9.5058e+012	15,000
F8	30	−1.0575e+004	−6.9827e+003	−4.5357e+003	1.3864e+003	15,000
	100	−1.5704e+004	−1.1507e+004	−8.6367e+003	2.1032e+003	15,000
	500	−3.9478e+004	−2.5673e+004	−1.9774e+004	4.8168e+003	15,000
	1000	−4.6339e+04	−3.4678e+04	−2.8024e+04	4143.2	15,000
F9	30	0	0	0	0	577
	100	0	0	0	0	779
	500	0	0	0	0	991.3333
	1000	0	0	0	0	1093
F10	30	8.8818e−016	8.8818e−016	8.8818e−016	0	15,000
	100	8.8818e−016	8.8818e−016	8.8818e−016	0	15,000
	500	8.8818e−016	8.8818e−016	8.8818e−016	0	15,000
	1000	8.8818e−016	8.8818e−016	8.8818e−016	0	15,000
F11	30	0	0	0	0	714.3333
	100	0	0	0	0	947.6667
	500	0	0	0	0	1261
	1000	0	0	0	0	1415

Table 6 (continued)

Function	Dimensions	Min	Mean	Max	STD	Mean FEs
F12	30	4.3576e−006	3.3494e−005	1.3230e−004	3.1907e−005	15,000
	100	0.1787	0.2815	0.4915	0.0756	15,000
	500	0.9127	5.0164e+009	1.9326e+010	8.4641e+009	15,000
	1000	3.4913e+010	3.7916e+010	4.0154e+010	1.2945e+009	15,000
F13	30	0.0138	0.3114	2.1749	0.4591	15,000
	100	8.0226	8.9178	9.9412	0.4664	15,000
	500	48.7718	2.2291e+009	3.3762e+010	8.4835e+009	15,000
	1000	99.3076	6.1961e+010	7.2374e+010	2.1104e+010	15,000

Table 7 Ranks of different algorithms

Algorithm	Dim	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	Sum
GSA	30	6	6	6	6	6	6	6	5	5	6	6	6	6	76
FA		5	5	5	5	5	4	4	4	6	5	4	4	3	59
CS		4	4	4	4	4	3	5	6	3	4	3	3	4	51
GWO		2	3	3	2	3	5	3	3	4	3	5	5	5	46
AOA		3	1	2	3	1	1	1	1	2	1	1	1	1	19
TP-AB		1	1	1	1	2	2	2	2	1	1	1	2	2	19
GSA	100	6	4	5	2	6	2	4	3	3	4	5	5	6	55
FA		4	6	6	5	5	1	5	2	6	6	4	6	5	61
CS		5	5	3	3	4	3	6	1	5	5	2	2	2	46
GWO		2	3	4	4	1	4	3	4	4	2	3	4	3	41
AOA		3	2	2	1	1	6	2	5	2	3	6	1	1	35
TP-AB		1	1	1	6	3	5	1	6	1	1	1	3	4	34
GSA	500	6	5	4	3	5	6	4	5	6	6	5	4	4	63
FA		5	6	6	4	6	5	6	4	5	5	6	5	5	68
CS		4	4	3	2	4	2	5	6	4	4	4	2	3	47
GWO		3	3	5	5	2	3	3	3	3	2	3	3	2	40
AOA		2	2	2	1	1	1	2	1	2	2	2	1	1	20
TP-AB		1	1	1	6	3	4	1	2	1	1	1	6	6	34
GSA	1000	5	5	4	3	4	5	4	5	4	6	6	4	4	59
FA		6	6	6	4	5	6	5	2	6	5	5	5	5	66
CS		4	4	3	2	3	3	3	3	5	4	3	2	3	42
GWO		3	3	5	5	2	2	2	4	3	3	4	3	2	41
AOA		2	2	2	1	1	1	1	1	2	2	2	1	1	19
TP-AB		1	1	1	6	6	4	6	6	1	1	1	6	6	46

Table 8 Overall ranking

Dim	30	100	500	1000	Total	Rank
GSA	76	55	63	59	253	V
FA	59	61	68	66	254	VI
CS	51	46	47	42	186	IV
GWO	46	41	40	41	168	III
AOA	**19**	35	**20**	**19**	**93**	I
TP-AB	**19**	**34**	34	46	133	II

dimensions and, for the 30 dimensions, both AOA and TP-AB are the joint best performers. AOA outperform other algorithms in the higher dimensions.

4 Constrained special functions and engineeringproblems

The benchmarks used for the analyses are grouped into four sets, Set 1, Set 2, Set 3 and Set 4. Initially, ten constrained special functions (Table 9) are considered for Set 1. The Cobb–Douglas model has two input variables; the units of capital (K) and, the units of labour (L) invested in the economy. The objective function is: Maximize Production = $300*K^{0.6}*L^{0.4}$. The labour cost is assumed as 85 units and the capital cost is 130 units. If the cost has to be kept below 1,00,000 then, the constraint can be expressed as, Total Cost $= 130*K + 85*L \leq 100{,}000$. The search range is taken as [0, 500] for both 'K' and 'L'.

Table 9 Set 1: constrained special functions

S. no.	Problem	No. of variables	No. of constraints	Optimal value
1	Cobb Douglas Utility Function (maximization problem)	2	1	139,541.6776
2	Himmelblau's Function (Himmelblau, 1972)	5	6	− 30,665.5398
3	Himmelblau's Function-II (Hu et al., 2003)	5	6	- 31,025.56142
4	Function, TP1 (Deb, 2000)	2	2	13:59,085
5	Function, TP4 (Deb, 2000)	8	6	7049:330,923
6	Function, TP5 (Deb, 2000)	7	4	680:6,300,573
7	Function, TP8 (Deb, 2000)	10	8	24.306209
8	Function, P5 (Mehta & Dasgupta, 2012)	2	2	0.095825
9	Function, P7 (Mehta & Dasgupta, 2012)	2	1	0.75
10	Function, P8 (Mehta & Dasgupta, 2012)	2	2	− 6961.81388

The problems, Test Function1 (TP1), TP5, TP6 and TP8 are taken from Deb (2000) paper whereas; Functions, P5, P7 and P8 are from Mehta and Dasgupta (2012).

Bayzidi et al. (2021) described and analysed several real-world engineering problems out of which 14 problems are considered in this work (Table 10) as a second set of constrained problems. In the third set, two more important problems are discussed and analysed (Table 11). Briefly, the Set 2 problems are discussed here.

(i) *Cantilever Beam***:** This benchmark deals with weight optimization of a five-section cantilever beam with a square hollow cross-section (Fig. 3). The beam is supported at the fixed end and loaded at its free end. The heights, h_i (equal to its widths) are the decision variables and the thickness is assumed to be constant and equal to 2/3.

The function to be minimized is: $\boldsymbol{f(x) = 0.0624*(h_1 + h_2 + h_3 + h_4 + h_5)}$.

The search span is: $0.01 \le x_i \le 100$; $i = 1$ to 5.

Table 10 Set 2: constrained engineering problems (Bayzidi et al., 2021)

S. no	Problem	No. of variables	No. of constraints	Optimal value
1	Cantilever beam	5	1	1.3399576
2	I-shaped beam	4	2	0.0130741
3	Three-bar truss	2	3	263.8958434
4	Tubular column	2	6	26.486361473
5	Speed reducer	7	11	2994.424466
6	Piston lever	4	4	8.41269832311
7	Corrugated bulkhead	4	6	6.8429580100808
8	Pressure vessel	4	4	6059.714335048436
9	Tension/compression spring (continuous domain)	3	4	0.01266051
10	Welded beam	4	7	1.724852308597366
11	Gear train	4	0	2.70085714e−12
12	Reinforced concrete beam	3	2	359.2080
13	Car side impact	11	10	22.84296954
14	Cantilever stepped beam (discrete)	10	11	63,893.43079587

Table 11 Set 3: two more constrained engineering problems

S. no.	Problem	No. of variables	No. of constraints	Optimal value
1	Tension/compression spring (discrete domain) (Deb & Goyal, 1997)	3	8	2.65855916
2	Cantilever stepped beam (continuous domain) (MathWorks Help Center)	10	11	63,408.9

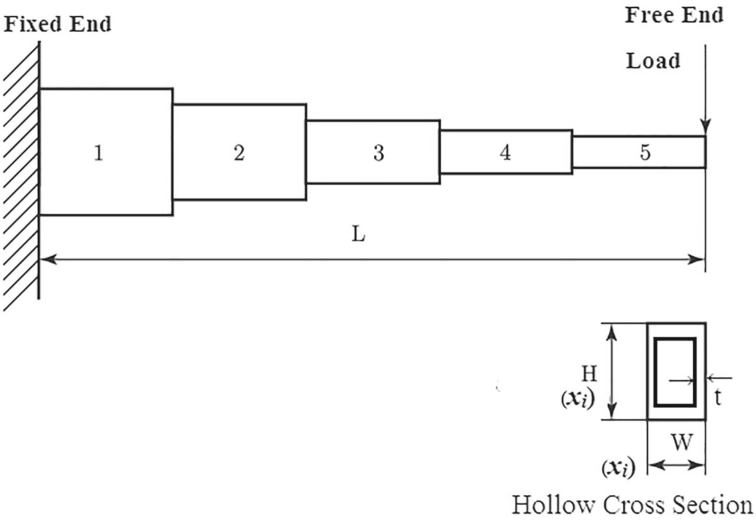
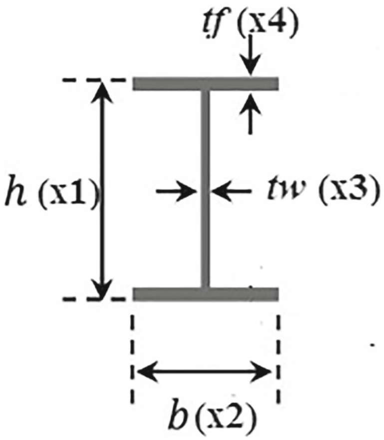


Fig. 3 Five-stepped cantilever beam

Fig. 4 I-shaped beam



(ii) *I-Shaped Beam*: The objective is to minimize vertical deflection (Fig. 4). The flange height, *h* (*x1*), beam width, *b* (*x2*), web thickness, *tw* (*x3*) and flange thickness, *tf* (*x4*) are the design variables and there are two constraints. The search spaces are;

$10 \le x1 \le 80$; $10 \le x2 \le 50$; $0.9 \le x3 \le 5$ and; $0.9 \le x4 \le 5$.

The formulated objective function is,

$$f(x) = \frac{5000}{tw(h-2*tf)^{4}/12 + \frac{b*tf^{3}}{6} + 2*b*tf(h-tf/2)^{2}}$$

(iii) *Three-Bar Truss*: In this problem, the volume of a 3-bar truss is to be minimized which is statically loaded (Fig. 5). The cross-section areas *A*1 (*x1*) and *A*2 (*x2*) are the variables and the search bounds are $2 \le x1 \le 14$, $0.2 \le x2 \le 0.8$.

The objective function is: minimize $\boldsymbol{f(x) = (2\,2*A1 + A2)*l}$; length, $\boldsymbol{l} = 100$ cm.

(iv) *Tubular Column*: This design problem involves designing a column to carry a compressive load at an optimized cost (Fig. 6). There are two continuous variables, the column

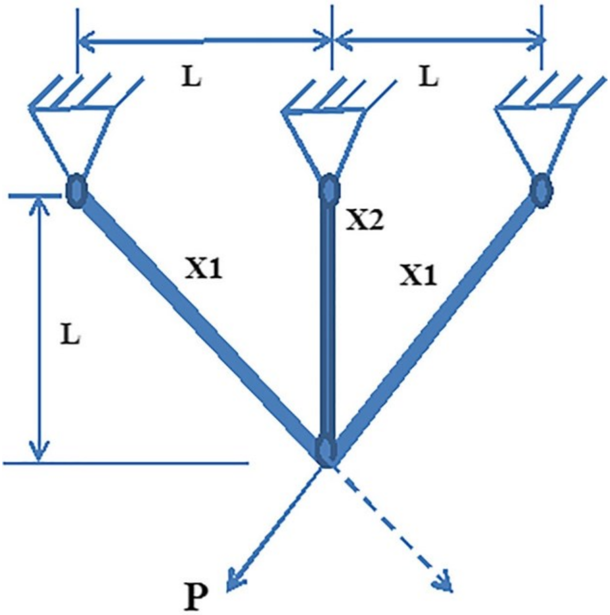


Fig. 5 Three-bar truss

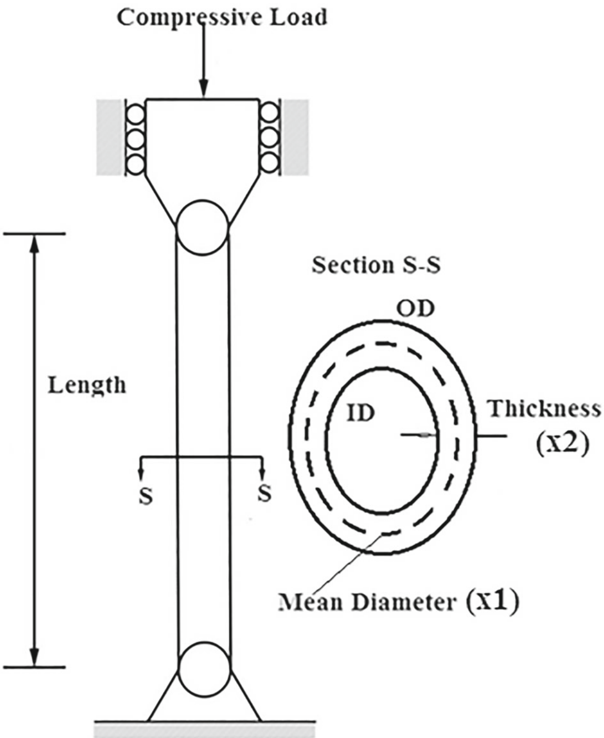


Fig. 6 Tubular column

mean diameter, *d* (*x1*) and the tube thickness, *t* (*x2*) along with six constraints. The search range is 2 ≤ *x1* ≤ 14 and 0.2 ≤ *x2* ≤ 0.8.

The function to be minimized is ***f*** **(*x*)** = 9.8*d*t + 2*d.

(v) *Speed Reducer*: The weight has to be optimized here which has 11 constraints (Fig. 7). The face width, *b* (*x1*), module, *m* (*x2*), number of teeth in the pinion, *z* (*x3*), length of the first shaft between bearings, *l*1(*x4*), length of the second shaft between bearings, *l*2(*x5*), the diameter of the first shaft, *d*1(*x6*), and the diameter of the second shaft, *d*2(*x7*) are the seven variables.

The cost function is: $f(x) = 0.7854*b*m^2*(3.3333*z^2 + 14.9334*z - 43.0934) - 1.508*b(d1^2 + d2^2) + 7.4777(d1^3 + d2^3) + 0.7854(l1*d1^2 + l2*d2^2)$. The bounds are: 2.6 ≤ *x1* ≤ 3.6, 0.7 ≤ *x2* ≤ 0.8, *x3* ∈ {17, 18, 19, …, 28}, 7.3 ≤ *x4, x5* ≤ 8.3, 2.9 ≤ *x6* ≤ 3.9, 5 ≤ *x7* ≤ 5.5.

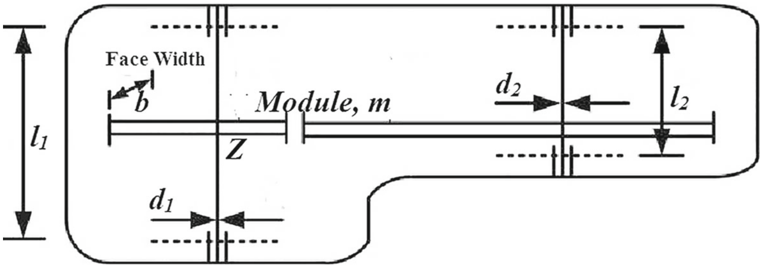


Fig. 7 Speed reducer

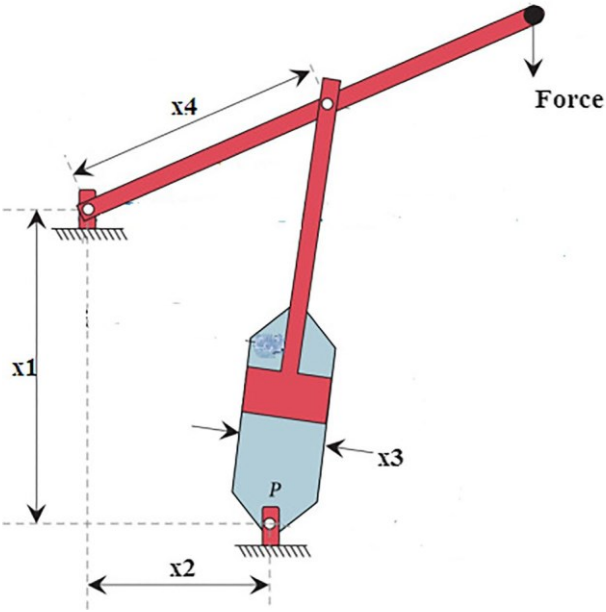


Fig. 8 Piston lever

(vi) *Piston Lever*: This problem has four location variables (Fig. 8); *H* (*x1*)*, B* (*x2*)*, D* (*x3*)*,* and *Y* (*x4*). The objective is to minimize the oil volume when the lever is lifted from 0° to 45°. That is, minimize; $f(x) = (1/4)*\pi*D^3(L2\text{-}L1)$; *L*1 and *L*2 will be computed from *H, B* and *Y*.

The search ranges are: 0.05 ≤ *x1*, *x2*, *x4* ≤ 500, 0.05 ≤ *x3* 120.

(vii) *Corrugated Bulkhead*: The weight of a corrugated bulkhead in a chemical tanker has to be optimized in this case (Fig. 9). The width, *w* (*x1*)*,* depth, *h* (*x2*)*,* length, *l* (*x3*)*,* and plate thickness, *t* (*x4*) are the four design variables. The bounds are: 0 ≤ *x1, x1, x3* ≤ 100, 0 ≤ *x4* ≤ 5.

The cost function is, minimize $f(x) = \frac{5.885*t(w+l)}{w+\sqrt{|l^2-h^2|}}$.

(viii) *Pressure Vessel*: The design involves minimization of the total cost (Fig. 10). The decision variables are; shell thickness, *ts* (*x1*)*,* head thickness, *th* (*x2*)*,* inner radius, *R* (*x3*)

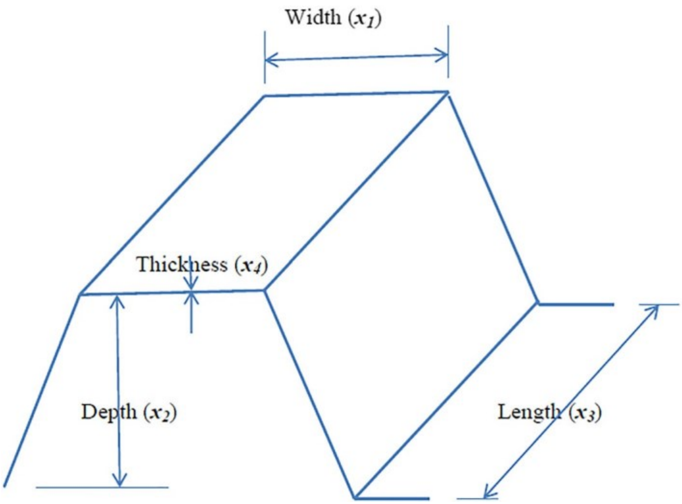


Fig. 9 Corrugated bulkhead

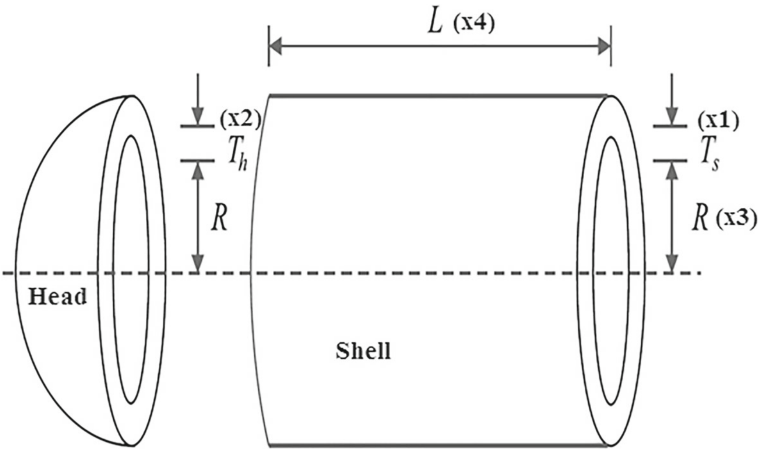


Fig. 10 Pressure vessel

and the length of the cylindrical section, L ($x4$). Also, the thickness is a multiple of 0.0625 inches for both the shell and head. The function to be minimized is:

Minimize: $f(x) = 0.6224*ts*R*L + 1.7781*th*R^2 + 3.1661*ts^2*L + 19.84*ts^2*R$. The search bounds are: $x1, x2 \in \{1*0.0625, 2*0.0625, 3*0.0625, ..., 1600*0.0625\}$, $10 \leq x3, x4 \leq 200$.

(ix) *Tension/Compression Spring (Continuous Domain)*: The problem is to design a spring with minimum weight (Fig. 11). It has 3 design variables subject to 4 constraints. The variables are mean coil diameter, D ($x1$,) wire diameter, d ($x2$) and the number of active coils, N ($x3$).

Minimize, $f(x) = (N + 2)*d*D^2$; $0.05 \leq x1 \leq 2$, $0.25 \leq x2 \leq 1.3$, $2 \leq x3 \leq 15$

(x) *Welded Beam*: The manufacturing cost of a welded beam is optimized (Fig. 12). The variables are; thickness of weld, h ($x1$), weld length, l ($x2$,) beam height, t ($x3$), and bar breadth, b ($x4$).

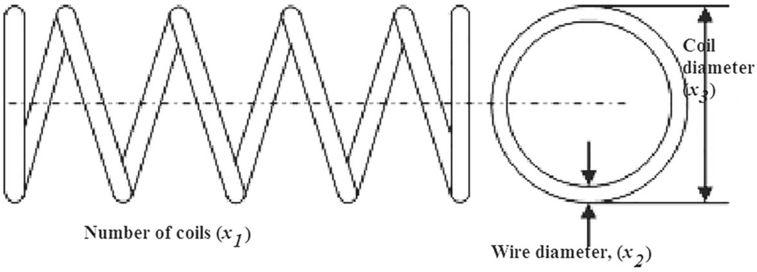


Fig. 11 Tension/compression spring

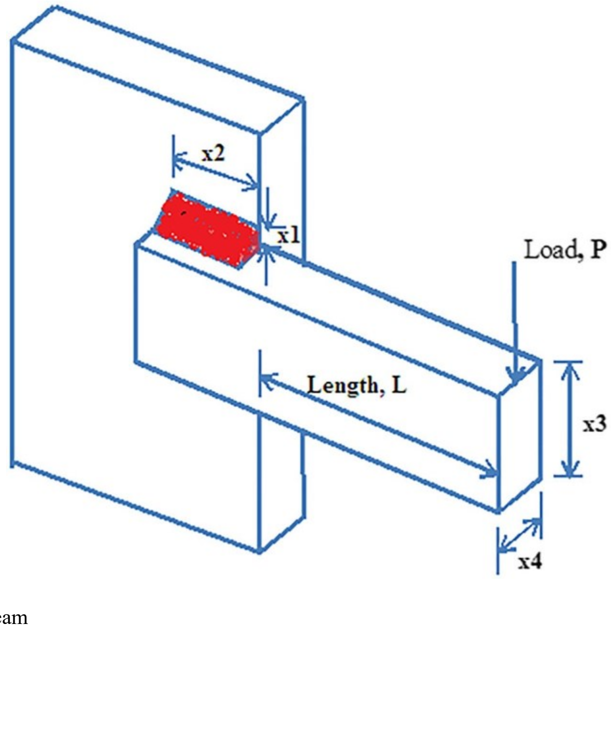


Fig. 12 Welded beam

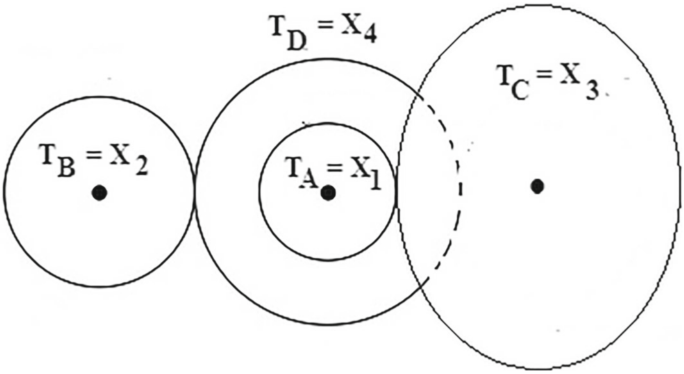


Fig. 13 Gear train

Minimize: $f(x) = 1.10471*h^2*l + 0.04811*t*b*(14.0 + l)$; $0.1 \le x1, x4 \le 2$, $0.1 \le x2, x3 \le 10$.

(xi) *Gear Train*: This problem is to design a compound gear train to achieve a specific velocity ratio (Fig. 13). This has 4 discrete integer variables (*x1, x2, x3* and *x4*) as they are the number of teeth in each gear which are in the range of [12, 13, 14, …, 60]. It does not have any constraints. The cost function is: Minimize, $f(x) = [(1/6.931) - (T_A * T_B)/(T_C * T_D)]^2$.

(xii) *RCC Beam*: The objective of this problem is to minimize the total cost of the simply supported reinforced cement concrete beam (Fig. 14). The reinforcement area, *A* (*x1*), beam width, *b* (*x2*) and, beam depth, *h* (*x3*) are to be evaluated. The first two variables are discrete whereas, the third one is continuous. There are two constraints imposed for the design. *x1*: [6, 6.16, 6.32, 6.6, 7, 7.11, 7.2, 7.8, 7.9, 8, 8.4]; *x2*: [28, 29, 30,..., 40]; and, $5 \le x3 \le 10$.

The cost function is: Minimize $f(x) = 2.9*A + 0.6*b*h$.

(xiii) *Car Side Impact*: During the car impact test, in addition to the safety requirements, the door weight is minimised (Fig. 15). It has eleven design variables: pillar inner thickness, *t*1 (*x1*), pillar reinforcement, *r*1 (*x2*), floor side inner, *s*1 (*x3*), cross member, *c* (*x4*), door beam, *b* (*x5*), door beltline reinforcement, *r*2 (*x6*), roof rail, *f* (*x7*), inner pillar material, *m* (*x8*), side inner, *s*2 (*x9*), barrier height, *h* (*x10*), and hitting position, *p* (*x11*). The cost function is: Minimize:

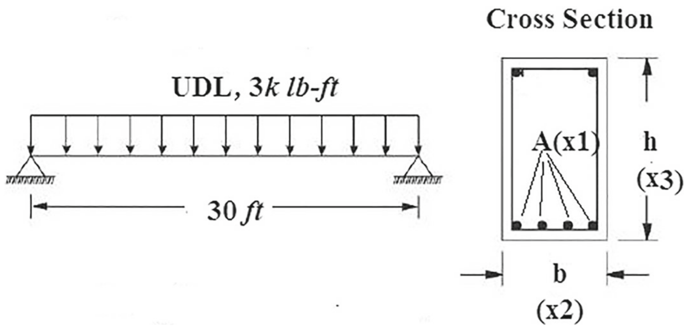
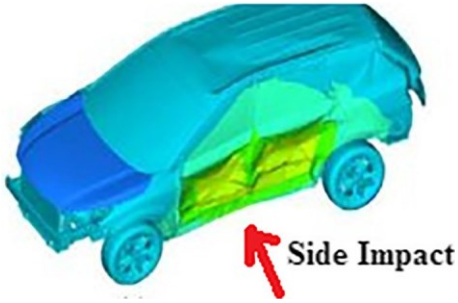


Fig. 14 RCC beam

Fig. 15 A model of car side impact problem



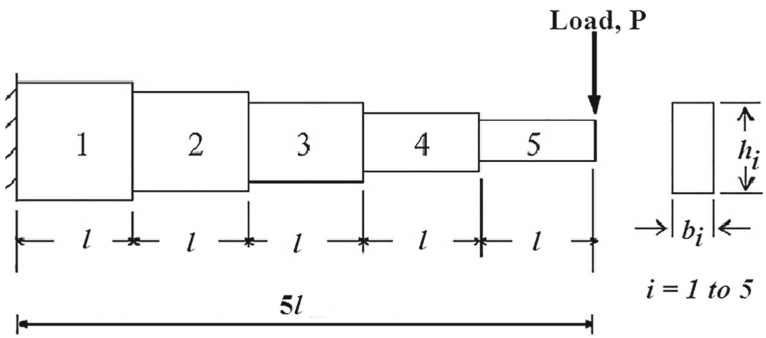


Fig. 16 Stepped cantilever beam

$\boldsymbol{f(x) = 1.98 + 4.90*t1 + 6.67*r1 + 6.98*s1 + 4.01*c + 1.78*b + 2.73*f}$; the variable ranges are: $0.5 \leq x1, x2, x3, x4, x5, x6, x7 \leq 1.5, x8, x9 \in \{0.192, 0.345\}, -30 \leq x10, x11 \leq +30$.

(xiv) *Stepped Cantilever Beam (Discrete)*: A five-stepped cantilever beam considered here is supported at one end and a load, P is applied at the free end (Fig. 16). The breadth (***b***), height (***h***) and length (***l***) are the decision variables and the step length '***l***' is the same for all steps and, the breadths and heights vary for each step. The volume of the beam has to be minimized subject to certain conditions. This problem is formulated with 10 variables and 11 constraints.

Volume, $\boldsymbol{V = L(b_i * h_i), i} = 1$ to 5.

Let, $x_1 = b_1, x_2 = h_1; x_3 = b_2, x_4 = h_2; \ldots; x_9 = b_5, x_{10} = h_5$. The first six ($x_1$ to x_6) are discrete variables (cross-section dimensions of the first three steps) and the remaining four are continuous. That is; *x1*: [1, 2, 3, 4, 5]; *x2, x4*: [45.0, 50.0, 55.0, 60.0]; *x3, x5*: [2.4, 2.6, 2.8, 3.1]; *x6*: [30, 31,..., 65]; $1 \leq x7, x9 \leq 5$, and $30 \leq x8, x10 \leq 65$.

The third set consists of two more engineering problems, the design simulation of a helical spring with discrete variables and the stepped cantilever beam with continuous variables (Table 11). The spring has 3 variables and 8 constraints. The variables are mean coil diameter, *D* (*x1*), wire diameter, *d* (*x2*) and the number of active coils, *N* (*x3*).

The objective is to minimize, $\boldsymbol{f(x) = 0.25*pi^2*d^2*D*(N+2)}$; 'N' is an integer between 1 to 32, the discrete variable 'd' has 42 non-equispaced values between 0.009 and 0.50 and, 'D' is continuous in the bounds $1 \leq D \leq 30$.

The stepped cantilever beam has 10 continuous variables with 11 constraints. This is the same problem as (xiv) of Set 2 except that all the variables are continuous. The bounds considered are; Lower Bound [1, 30, 2.4, 45, 2.4, 45, 1, 30, 1, 30]; Upper Bound [5, 65, 3.1, 60, 3.1, 60, 5, 65, 5, 65] for the 10 variables respectively.

5 Results and discussion

The codes are generated in MATLAB 2011a and run on a desktop i5 PC with 4 GB RAM. 30 trials are conducted for each problem with different population size (PS) and number of iterations (IT). The parameters considered are indicated in each table appropriately.

The proposed algorithm is run with and without a tuning parameter and the "*Min*" and "*Mean*" values are tabulated. The tuning parameter ***"$\alpha = 1$"*** for all the problems except the pressure vessel in which it is taken as 2. The better results are shown in bold letters in all tables.

5.1 Constrained special functions

The codes are run for this set of special functions for a specified number of population sizes and iterations. The statistical results are presented in Table 12 and 13 and the obtained results are compared with a few popular algorithms (Table 14).

For the Cobb–Douglas utility maximization problem, since no other results are available, only the obtained results are reported. The cost obtained (−139,541.1891) in both cases is very close to the optimal value of 139,541.6776. When the "*Min.*" values are considered,

TP-AB performs better in all the cases except "Function TP8" and "Function P8". For the "Mean" values; Deb's algorithm yields better results in "Functions TP4, TP5 and TP8". However, the number of function evaluations used by Deb for getting these results is very high; 3,20,000, 3,50,000 and 3,50,000 respectively for each problem. For the TP4 function, the best values are obtained by TP-AB after many simulations of 30 trials each. In the case of P5 and P7 functions, TP-AB performs well for the "Min" values for less number of FEs. However, for the P8 function, Mehta and Dasgupta report a better "Min." value for this function for the same number of FEs. The "Mean" values could not be reported for TP-AB as it yields higher values due to the violation of the constraint(s) in a few iterations. The "Mean" values are not reported by the authors and hence left blank in the table. The results are also reported for the second version of Himmelblau's Function-II (Hu et al., 2003) with the known results for the bounds used and constraints. TP-AB performs better for this version also. TP-AB (no tuning) performs better in more problems when compared with TP-AB (with tuning) for this set of functions.

The real-world application of the "Function, TP4" is a "Heat Exchanger Design" problem. Hu et al. (2023) analysed this problem using their "Multi-strategy boosted Chameleon-inspired Optimization Algorithm (MCSA)" and compared the results with several recent algorithms. Thirty trials were conducted with a maximum of 50,000 FEs. This is regarded as a difficult problem since the constraints are interrelated and binding. The bounds are also widely separated [$100 \leq x1 \leq 10{,}000$, $1000 \leq x2, x3 \leq 10{,}000$, $10 \leq xi \leq 1000$ ($i = 4, 5, 6, 7, 8$)] making it difficult for any population-based algorithm to converge towards the optimal value which is 7049:330,923. MCSA reported a cost value of 7273.443 with variables [100.000, 1879.955, 5293.487, 120.210, 288.261, 273.171, 231.949, 388.261]. When TP-AB (with tuning) is run for the same 50,000 FEs; a cost

Table 12 Results of Set 1: constrained special functions (no tuning)

No	Problem	Min	Mean
1	Cobb–Douglas	−139,541.18914658113499172031879425	−139,540.74731565261026844382286072
X = [461.54890593202361515068332664669, 470.57226151336220709708868525922]			
2	Himmelblau's Function	−30,665.53867178306973073631525039 7	−30,665.53856993323279311880469322 2
X = [78.00000000000396482846554135903 7, 33.0, 29.99525602568176907425367971882 2, 45.0, 36.77581290578703487881284672766 9]			
3	Himmelblau's Function−II	−31,025.56024249794063507579267025	−31,025.54636139738431666046380996 7
X = [78.0, 33.0, 27.07099710517604762571863830089 6, 4 .0, 44.96924255010550552924541989341 4]			
4	Function, TP1	13.59084169474095915575162507593 6	13.59356536222869316077321855118 5
X = [2.24682619458242527343827532604 34, 2.38187011134724713201649137772 62]			
5	Function, TP4	7105.81693014700704225106164813 04	8359.26489474997288198210299015 05
X = [585.88702287981732297339476644993, 1005.0881674645683006019680760801, 5514.8417398026213049888610839844, 182.47654694642662320802628528327, 279.40637043886539458981133066118, 217.31097775835476682004809845239, 303.06969230500084222512668929994, 379.40635502636939691001316532493]			
6	Function, TP5	680.63156102137247671635122969747	680.71010130230774848314467817545
X = [2.32786318814271853128161637869 2, 1.95224147145824877824793475156 18, − 0.47379022488758049114210280094994, 4.36360206337381306695988314459 1, − 0.62100377836629039318694367466378, 1.03080376679572527365280620870 18, 1.58696139551072357143368662946 16]			
7	Function, TP8	25.37385191417217455978061479982	29.31441014508602904697909252718 1
X = [2.18203140979412024691441729373 76, 2.33636292037366821716659615049 14, 8.76470958597910509979556081816 55, 5.07162257045567432811594699160 2, 0.96938739779637050908434048324125, 1.39384949382237399717610060179 14, 1.34275970415147982706116636109 07, 9.84647336233922487735981121659 28, 8.06264153199695954299386357888 58, 7.76486752351137621275256606168 11]			

Table 12 (continued)

No	Problem	Min	Mean
8	Function, P5	−0.095825041418035855622648000462505	−0.095824334967236399207557440149685
X = [1.2279713520865258580272438848624, 4.2453733660166292906978924293071]			
9	Function, P7	0.75000345245841260055641441795160	0.77197085855291236455855141684879
X = [−0.70589942121796067642236494066631, 0.49829345272105457498312830466602]			
10	Function, P8	−6961.7949261931025830563157796860	–
X = [14.095007828693153939525473106187, 0.84297764294916754934661184961442]			

No	Max	SD	PS*It
1	−139,533.49179176311008632183074951	1.4205554115588354679999838481308	5 × 2000
2	−30,665.537468068076123017817735672	0.00028501002882999322638607475610684	5 × 1500
3	−31,025.187317002753843553364276886	0.067908224917406806442343736307521	5 × 1500
4	13.669736187511960068263761058915	0.014390344646147973026173438881 87	5 × 500
5	13,198.327056275011273100972175598	1428.8269871019624588370788844244	5 × 10,000
6	681.07289185388742680515861138701	0.10190623992291888977312908082 24	5 × 8000
7	55.196846159783966356826567789 54	6.5958854001139410883070013369434	5 × 5000
8	−0.095803925445478721467651439525071	0.0000038547692326700641669737050498679	5 × 150
9	0.99999999999999995559107901499373 8	0.062011290080469863572609767743415	5 × 150
10	–	–	5 × 560

Table 13 Results of Set 1: constrained special functions (with tuning)

No	Problem	Min	Mean
1	Cobb–Douglas	−139,541.18911382160149514675140381	−139,541.06767416873481124639511108
X = [461.54197083743036955638672225177, 470.58286745159426800455548800528]			
2	Himmelblau's Function	−30,665.538671765545586822554469109	−30,664.985758020251523703336715698
X = [78.0, 33.0, 29.995256025794756027380572049879, 45.0, 36.775812905502881733355025062338]			
3	Himmelblau's Function−II	−31,025.560242466228373814374208 45	−31,025.488293497124686837196350098
X = [78.0, 33.0, 27.070997105348435951555075007491, 4 .9999999994381809642618463840 34, 44.969242549824841148620180319995]			
4	unction, TP1	13.590841703571683751761156599969	13.609253036608816245234265807085
X = [2.2468263804271351169461468089139, 2.3818735717325036915781311108731]			
5	Function, TP4	7051.5869454635103465989232063293	8928.8005707804240955738350749016
X = [642.14789226293282808910589665174, 1326.7114648572830901684937998652, 5082.7275883432948830886743962765, 187.04260035990481014778197277337, 296.69128178138248586037661880255, 212.95738900444274577239411883056, 290.35131454624706748290918767452, 396.69110893759790315016289241612]			
6	Function, TP5	680.63188683062537620571674779058	680.74887997389180327445501461625
[2.3312283383209004838931832637172, 1.9510214798928720014714599528816, − 0.46187032206071604312214162746386, 4.3652011797544627569322983617894, − 0.61909852737152915036489275735221, 1.0334627814669596901353543216828, 1.5904017518745869885776755836559]			
7	Function, TP8	25.382865152539562103584103169851	27.351210912291495702675092616118
X = [2.1840201089686606117368228296982, 2.3329088333996614856857831910 02, 8.7622150837558177016717309015803, 5.0714753313435831927336039370857, 0.97408710388437558247431979907 56, 1.4064319064139227943854848490446, 1.3455251347970458297709228645544, 9.8484302960177281249798397766426, 8.0382196992646122168935107765719, 7.6923780338670590950300720578525]			

Table 13 (continued)

No	Problem	Min	Mean
8	Function, P5	−0.095825041418035841744860192648048	−0.095824226867466402723927387796721
X = [1.2279713538699887909189101264928, 4.2453733674848015411384949402418]			
9	tion, P7	0.75000058891757859225180027351598	0.75842809850604619992253674354288
X = [0.70731898309785801171045704904827 0.50029964471996757335858774240478]			
10	Function, P8	−6961.8069599080035914084874093533	–
X = [14.095002985093152858553366968408, 0.84296693421782464827884950864245]			

No	Max	SD	**PS*It**
1	−139,540.24236588340136222541332245	0.2575750511696743294542955027282	5 × 2000
2	−30,656.045765637602016795426607132	1.9174280574021387302252605877584	5 × 1500
3	−31,025.104442477913835318759083748	0.13253615971549684293684379099432	5 × 1500
4	13.992417516434111135481543897185	0.075055773525795094203161283985537	5 × 500
5	18,231.852764200273668393492698669	2988.390046422426621575141325593	5 × 10,000
6	681.09021460050462337676435709	0.11944963757261693737543595261741	5 × 8000
7	34.493206601413483269880089210346	2.1944566711017721161169902188703	5 × 5000
8	−0.095801625499109005490439017194149	0.0000042724461756200131425120249117899	5 × 150
9	0.99246285060491723495346150230034	0.044208215304290893188365885180247	5 × 150
10	–	–	5 × 560

Table 14 Set 1: comparison of results with known algorithms [30 Trials]

	Problem S. no	Known best [Authors, FEs]	Mean	TP-AB (no tuning)		PSxIt	TP-AB (with tuning, a = 1)	
				Best*	Mean*		Best*	**Mean***
1	Cobb–Douglas	N/A	N/A	**−139,541.1891**	−139,540.7473	5 ×	**−139,541.1891**	**−139,541.0677**
2	Himmelblau's Function	−30,665.537 [Deb,	−30,665:535	−30,665.53867178	**−30,665.53856993**	5 × 2000	**−30,665.53867177**	−30,664.98575802
3	Himmelblau's Function-II	−31,0[illegible] (Fesanghary et al., 2008)	–	**−31,025.56024250**	**−31,025.54636140**	5 × 1500 1500	−31,025.56024247	−31,025.48829350
4	Function, TP1	13.59085 [Deb, 50 × 50]	13.61673	**13.59084169**	**13.59356536**	5 ×	13.59084170	13.60925304
5	Function, TP4	7060.221 [Deb, 80 × 4000]	**7220.026**	7105.81693	**8359.26489**	5 × 500	**7051.58695**	8928.80057
6	Function, TP5	680.634460 [Deb, 70 × 5000]	**680.641724**	**680.63156102**	**680.71010130**	5 × 10,000	680.63188683	680.74887997
7	Function, TP8	24.37248 [Deb, 100 × 3500]	**24.40940**	**25.373852**	29.314410	5 × 8000	25.382865	**27.351211**
8	Function, P5	0.095825 (Mehta & Dasgupta, 2039)	N/A	**−0.0958250**	−0.0958243	5 × 5000	**−0.0958250**	−0.0958242
9	Function, P7	0.75 (Mehta & Dasgupta, 1935]	N/A	0.750003	0.771971	5 × 150	**0.750001**	**0.758428**
10	Function, P8	**− 6961.81387** (Mehta & Dasgupta, 5623)	N/A	−6961.794926	–	5 × 150 560	−6961.806960	–

*The values in brackets show the population size *number of iterations used, in all tables of 7067.752 and variables [721.625, 1432.648, 4913.479, 192.816, 303.461, 207.184, 289.355, 403.461] with constraints set of [− 2.8711e−07, − 1.0226e−07, − 1.0497e−06, − 1.3867e−04, − 1.1257e−02, − 3.9031e−02] are reported. TP-AB without tuning reports a slightly higher cost value, 7068.782, variables [737.641, 1409.515, 4921.625, 193.90884119287710518619860522449, 303.13498033033158662874484434724, 206.09115619329122637282125651836, 290.774, 403.135] and constraints [− 6.5346e−09, − 2.9184e−10, − 5.9171e−10, − 2.4603e−04, − 2.9182e−03, − 4.8074e−04]. That is, the performance of TP-AB is better than MCSA for this difficult problem for the same number of FEs satisfying all the constraints.

5.2 Constrained engineering problems

Tables 15 and 16 show the results of the 14 constrained real-world problems (Set 2). This set of problems is a mix of continuous and discrete variables. Out of the 14 problems, the following benchmarks have discrete or discrete and continuous variables:

Pressure Vessel: First two variables are discrete.

Gear Train: All four variables are discrete.

Reinforced Cement Concrete Beam: First two variables are discrete.

Spring (Discrete): First two variables are discrete.

Cantilever Stepped Beam (Discrete): First six variables are discrete.

The results of other algorithms are taken from the SNS paper (Bayzidi et al., 2021). If two or more algorithms report the same cost value, only the algorithm that uses a minimum number of FEs is considered. Several popular and better-performing algorithms were used by Bayzidi et al., 2021 to establish the performance of their Social Network Search (SNS) algorithm which include: AOS-Atomic Orbital Search (Azizi et al., 2021); WCA-Water Cycle Algorithm (Eskandar et al., 2012); CGO-Chaos Game Optimization (Talatahari & Azizi, 2020); MCEO- Multilevel Cross Entropy Optimizer (MiarNaeimi et al., 2018); CS-Cuckoo search (Yang & Deb, 2010); WOA- Whale Optimization Algorithm (Mirjalili & Lewis, 2016) and WSA- Water Strider Algorithm (Kaveh & Eslamlou, 2020).

Table 17 shows the comparison of results for the "*Min*" values and the results for the "*Mean*" values are presented in Table 18.

For the "Best" values, TP-AB is a better performer yielding the best results in 11 of the 14 problems in both "With and Without Tuning" cases. SNS algorithm accounts for 6 better results. Other algorithms; WCA, MCEO, CS and AOS report better results in one case each.

For the "*Mean*" values, the performance of TP-AB is moderate. It accounts for only 3 best results out of 13 problems. SNS is the better performer here accountable for better results in 4 cases. AOS, CGO, WOA WSA and CS report best "*Mean*" values in one case each.

For all the problems except the "Cantilever Stepped Beam (Discrete)", the constraints are satisfied. In the "Stepped Cantilever (Discrete)" problem, the sixth constraint value is reported as a positive number (**4.7155e−02)** which violates the constraints condition. When this value multiplied by the "Penalty, 10^3" is subtracted from the obtained cost, the real cost is obtained.

That is, Cost 63,940.585524715192150324583053589 (10^3*4.7155e−02) − 63,893.430795872212911490350961685 which is given in the next row. However, it is to be noted that this cost is obtained after violating constraint number six.

The SNS algorithm reports a cost of 63,893.4307958715 which is close to the value obtained above. However, when the results given in the SNS paper are analysed, the obtained constraints set is, G = [− 1.4881e−06, − 2.3780e−06, − 1.5385e + 02, − 1.2034e + 03,

Table 15 Results of Set 2: constrained engineering problems (no tuning)

No	Problem	Min	Mean
1	Cantilever Beam	1.3399565533525290561556175816804	1.3399584283390073569108835727093
X = [6.0157643269530911567244402249344, 5.3108520696457199861129083728883, 4.4943708510850290949178997834679, 3.5014630513617852614061121130362, 2.151212414937213157628548287903]			
G = [−6.0436e−09]			
2	I−shaped beam	0.013074118905223334682896840774902	0.013074761168925133278384542734329
X = [80.0, 50.0, 0.9, 2.3217922606924643584521384928717]			
G = [0, − 1.5702]			
3	Three−bar truss	263.89584337746543951652711257339	268.94834729636841075262054800987
X = [0.78867598887076773017668074317044, 0.40824587421655034180645316155278] G = [−3.4917e−12, − 1.4641e + 00, − 5.3590e−01]			
4	Tubular column	26.486361472447814691122403019108	26.486361472450084875163156539202
X = [5.4521807362239060879005592141766, 0.29162642929940890690332366830262]			
G = [0, 0, − 0.6332, − 0.6106, − 0.6332, − 0.3185]			
5	Speed reducer	2994.4244657567364811256993561983	3006.525760085181900649331510067
X = [3.5, 0.7, 17.0, 7.3, 7.7153199114782466949691297486424, 3.350540949105893506754227928468, 5.2866544649802218458489733166061]			
G = [−2.1550, − 98.1350, − 1.9251, − 18.3099, − 0.0000, 0, − 28.1000, 0, − 7.0000, − 0.3742, − 0.0000]			
6	Piston lever	8.4126983231064489388018046156503	8.4127020184012941683704411843792
X = [0.0500000000000003819167204710538499, 2.0415135899181171552640989830252, 4.0830271798362200996734827640466, 120.0] G = [−9.3132e−10, − 6.0000e + 05, − 1.1719e + 02, − 7.1054e−15]			
7	Corrugated bulkhead	6.842958010080779196471212344477	6.8429580198659172296515862399247
X = [57.692307692307692307692307692308, 34.1476203486743870030295511242, 57.692307692307692307692307692308, 1.05]			
G = [−240.6946, 0.0000, − 0.0000, − 0.0000, 0, − 23.5447]			
8	Pressure vessel	6059.7143350484611801221035420895	6366.0204421799053307040594518185
X = [0.8125, 0.4375, 42.098445595854919076828082324937, 176.63659584244194888924539554864]			

Table 15 (continued)

No	Problem	Min	Mean
G = [0, − 0.0359, − 0.0000, − 63.3634]			
9	Tension/compression spring (continuous)	0.012666076883382516513054660833859	0.013153040852535168719894542732618
X = [0.051764246615505479787167075755586, 0.35852123267507962633970919341664,			
11.184614729736463090148390620016] G = [−2.8498e−05, − 1.7898e−05, − 4.0571e + 00, − 7.2648e−01]			
10	Welded beam	1.7248523086028759720989000925329	1.7249382895883313970841754780849
X = [0.20572963978470074075666218504921, 3.4704886656610494100050345878117, 9.0366239103707322044556349283084,			
0.20572963978620054104062830901967] G = [−2.6321e−08, − 1.0462e−07, − 1.4998e−12, − 3.3907e + 00, − 8.0730e−02, − 2.3554e−01, − 1.6314e−08]			
11	Gear train	0.0000000000027008571488865133510618295276 27	0.0000000010375341991661390031385232124829
X = [49, 19, 16, 43]			
No constraints			
12	Reinforced concrete beam	359.2080	360.33926687022699297813232988119
X = [6.3200, 34.0000, 8.5000]			
G = [0, − 0.2241]			
13	Car side impact	22.842972911474802799602912273258	22.981409570086889004869590280578
X = [0.50000000001910460678544723123196, 1.1166237513462768937699820526177, 0.5, 1.3017686987308378920857876437367, 0.50000000008376155324896217280184, 1.5, 0.50000000287244161789601548662176, 0.71092426427884225503817106073257, 0.00030714256594492595452139394041069, − 19.515354335250712836113962111995, 0.000027572348736335856216397854900 89]			
G = [−6.1702e−01, − 9.2690e−02, − 1.0063e−01, − 3.4181e−02, − 4.2757e + 00, − 7.2776e + 00, − 8.6929e−10, − 4.5892e−12, − 9.6457e−01, − 1.6637e−01]			
14	Cantilever stepped beam (Discrete)	63,940.585524715192150324583053589	66,061.911281781314755789935588837
		63,893.430795872212911490350961685	–
X = [3.0, 60.0, 3.1, 55.0, 2.6, 50.0, 2.2045556915418584864596596162301, 44.091113830837251441607804736122, 1.7497570119380985165236097600427, 34.995140238761408113532525021583]			

Table 15 (continued)

No	Problem	Min	Mean
G = [−1.8172e−09, − 2.8376e−10, − 1.5385e + 02, − 1.2034e + 03, − 1.1111e + 02, **4.7155e−02**, − 3.1974e−13, 3.5527e−14, − 7.6923e−01, − 2.2581e + 00, 0]			

No	Max	SD	PS*It
1	1.3399625431634891725707348086871	0.000001605125341295003802197629075299	10 × 5000
2	0.013086594173798100915195874449637	0.0000022948460775481232742857747031184	5 × 360
3	282.84271247461900976033774484194	8.5218520594414819413486839039251	5 × 10,000
4	26.486361472483061163529782788828	0.0000000000066181359951836678015183704802052	10 × 750
5	3141.019025073047941987169906497	29.765193762810703503873810404912	5 × 1000
6	8.4127820509925470560119720175862	0.000015854558376347181432655664146125	5 × 500
7	6.8429581296382666621980206400622	0.000000027808917762723318779599534947104	10 × 1000
8	7544.4925179250703877187334001064	474.4038581463772743518347851932	
9	0.015192618351934328649321948034867	0.00066986932383081428891286046578557	5 × 900
10	1.7257568703007288846151823236141	0.00019981081626932833375694109623311	**10 × 1500**
11	0.000000003824836866277994244321619492482	0.00000000093939488063877743171087118253344	10 × 1250
No Constraints			
12	362.63400000000001455191522836685	1.5123811414479211290284865754074	5 × 100
13	24.183807228547788525929718161933	0.33126731123996738803683115293097	5 × 2000
14	72,592.00222516800567973405122757	3015.6665037791522081533912569284	5 × 2000
	–	–	

Table 16 Results of Set 2: constrained engineering problems (with tuning)

No	Problem	Min	Mean
1	Cantilever Beam	1.3399564656693445652280161084491	1.3399619524765717937242470725323
X = [6.0154496467444857898954069241881, 5.3094069244844481758605070353951, 4.495819960522809743963534856448, 3.5007490713326223641388423857279, 2.1522357057192356144526002026396]			
G = [−1.3440e−08]			
2	I−shaped beam	0.013074118905223334682896840774902	0.013074353065533457496361080529823
X = [80.0, 50.0, 0.9,2.32179226069246435845213 4928717]			
G = [0, − 1.5702]			
3	Three−bar truss	263.89584337929170487768715247512	267.68527634624967959098285064101
X = [0.78867355220061774456752345940913, 0.40825276617875899676235462720797] G = [−7.4512e−12, − 1.4641e + 00, − 5.3590e−01]			
4	Tubular column	26.486361472447814691122403019108	26.486361472473781475400755880401
X = [5.4521807362239060879005592141766, 0.29162642929940890690332366830262]			
G = [0, 0, − 0.6332, − 0.6106, − 0.6332, − 0.3185]			
5	Speed reducer	2994.4244657572799042100086808205	2999.7966171012249105842784047127
X = [3.5000000000003548272786702000303, 0.7, 17.000000000000408562073062057607, 7.3, 7.7153199114797841318136306654196, 3.3505409491059601201357054378605, 5.2866544649806677114156627794728]			
G = [− 2.1550e + 00, − 9.8135e + 01, − 1.9251e + 00, − 1.8310e + 01, − 6.7985e − 11, − 2.1475e − 10, − 2.8100e + 01, − 5.0715e − 13, − 7.0000e + 00, − 3.7419e − 01, − 1.0494e − 12]			
6	Piston lever	8.4126983231272998153826847556047	8.4160349650843873092753710807301
X = [0.05, 2.0415135899206351410839488380589, 4.0830271798387967052690328273457, 120.0]			
G = [− 2.1788e − 06, − 6.0000e + 05, − 1.1719e + 02, − 1.2368e − 12]			
7	Corrugated bulkhead	6.8429580105405225509684896678664	6.8430300163287052228611173632089
X = [57.692307691576530714883119799197, 34.147620343983483337524376111105, 57.692307662695419878673419589177, 1.0500000000037816860753991932143]			

Table 16 (continued)

No	Problem	Min	Mean
G = [− 2.4069e + 02, − 8.7297e − 08, − 1.5188e − 11, − 4.6573e − 10, − 3.7816e − 12, − 2.3545e + 01]			
8	Pressure vessel	6059.7143350489850490703247487545	6089.8141894956643227487802505493
X = [0.8125, 0.4375, 42.098445595852169276440690737218, 176.6365958424826771988591644913]			
G = [− 5.3069e − 14, − 3.5881e − 02, − 5.0990e − 08, − 6.3363e + 01]			
9	Tension/compression spring (continuous)	0.012665311959197685098832408812086	0.01285010530546863358192233306454
X = [0.051661833512531868395356582368549, 0.35606224615212012096776561520528, 11.327550388376881329577372525819]			
G = [− 1.8824e − 06, − 1.8594e − 06, − 4.0525e + 00, − 7.2818e − 01]			
10	Welded beam	1.7248523095169323582354081736412	1.7248677262057536818673497691634
X = [0.20572963980751046708661533557461, 3.4704886693777572403973863401916, 9.0366239083929826847452204674482, 0.20572963988018633241239285780466]			
G = [− 1.0805e − 05, − 6.7830e − 07, − 7.2676e − 11, − 3.3907e + 00, − 8.0730e − 02, − 2.3554e − 01, − 7.3761e − 06]			
11	Gear train	0.000000000002700857148886513351061829527627	0.00000000045719424504134971327783514457026
X = [43, 19, 16, 49]			
No constraints			
12	Reinforced concrete beam	359.2080	360.12060000580925134272547438741
X = [6.3200, 34.0000, 8.5000]			
G = [0, − 0.2241]			
13	Car side impact	22.842995170222621936773066408932	23.008457385352659940735975396819
X = [0.50000057922109453212300422819681, 1.1162390753010220656449291709578, 0.50000430016527364074363504187204, 1.3024055595166994514499947399599, 0.50000078144149917580563169394736, 1.5, 0.5, 0.6131427881882459685058961440518, 0.65447689495107042034760524984449, − 19.583956686382084200204189983197, 0.057115698892089186711196191481577]			
G = [− 6.5487e − 01, − 7.4241e − 02, − 1.0224e − 01, − 4.6883e − 02, − 3.7171e + 00, − 6.1254e + 00, − 7.4841e − 07, − 7.5383e − 08, − 9.6575e − 01, − 2.9748e − 01]			

Table 16 (continued)

No	Problem	Min	Mean
14	Cantilever stepped beam (Discrete)	63,940.5855252556939376518130302 43	66,791.7837287355650914832949638 37
		63,893.4307964014005847275257110 60	–

X = [3.0, 60.0, 3.1, 55.0, 2.6, 50.0, 2.2045556915506772099888621596619, 44.091113830775562121289112837985, 1.7497570122247261270587159742718, 34.995140235908756665139662800357]

G = [− 1.2718e − 08, − 1.7111e − 08, − 1.5385e + 02, − 1.2034e + 03, − 1.1111e + 02, **4.7155e − 02**, − 4.9068e − 09, − 1.0795e − 10, − 7.6923e − 01, − 2.2581e + 00, 0]

No	Max	SD	PS*It
1	1.3399754653321582775760134609 3	0.0000043341852988437834883796162532832	10 × 5000
2	0.013076073978803061728659606899328	0.0000004538592983407298128257720010864 1	5 × 360
3	282.84271247461900976033774484194	7.7082781917268921034747108933516	5 × 10,000
4	26.486361472668981775768770603463	0.0000000000518566289253349701066951510463 44	10 × 750
5	3051.4248097353652156016323715448	15.285597961766001162686734460294	5 × 1000
6	8.4684457804075545794830759405158	0.010929035787079596081516896788344	5 × 500
7	6.8441840069909334332010075740982	0.00022619429995553120768536636830959	10 × 1000
8	6786.7829008133676325087435543537	132.38824569621397131413687020 54	
9	0.014377507289041215748914837035954	0.00036626240928083644669832619200633	5 × 900
10	1.7249314809935545333985373872565	0.00020025048288782693549879565519944	**10 × 1500**
11	0.000000002357640658024884377214578783403 2	0.0000000005819962181549071289653164946802 9	10 × 1250
No Constraints			
12	362.250	1.4178506424174766031853778258665	5 × 100
13	23.597766705575750734169560018927	0.22957433830772541738518555121118	5 × 2000

Table 16 (continued)

No	Max	SD	PS*It
14	73,790.5525291320082033053040504460 –	2981.929178753952783154090866446 –	5 × 2000

Table 17 Comparison of results of set 2: constrained engineering problems; best values

S. no.	Problem	Best	Algorithm (FEs)	TP-AB (no tuning)	PS * IT	TP-AB (with tuning)
1	Cantilever beam	1.339957	AOS (100,000)	1.339957	10 × 5000	**1.339956**
2	I-shaped beam	**0.0130741**	SNS (3600)	**0.0130741**	5 × 360	**0.0130741**
3	Three-bar truss	**263.895843**	WCA (5250)	**263.895843**	5 × 10,000	**263.895843**
4	Tubular column	**26.48636147**	SNS (1250)	**26.48636147**	10 × 750	**26.48636147**
5	Speed reducer	2994.443649	CGO (100,000)	**2994.424466**	5 × 1000	**2994.424466**
6	Piston lever	8.412698349	SNS (5000)	**8.412698323**	5 × 500	**8.412698323**
7	Corrugated bulkhead	**6.84295801**	AOS (100,000)	**6.842958010**	10 × 1000	**6.842958010**
8	Pressure vessel	**6059.714335**	SNS (6000)	**6059.714335**	10 × 1500	**6059.714335**
9	Tension/compression spring (continuous)	**0.01266051**	MCEO (2000)	0.012666077	5 × 900	0.012665312
10	Welded beam	**1.724852**	SNS (9000)	**1.724852**	10 × 1500	**1.724852**
11	Gear train	**2.700857E−12**	SNS (25,000)	**2.700857e − 12**	10 × 1250	**2.700857e − 12**
12	Reinforced concrete beam	**359.2080**	SNS (1000)	**359.2080**	5 × 100	**359.2080**
13	Car side impact	**22.84294**	CS (20,000)	22.842973	5 × 2000	22.842995
14	Cantilever stepped beam (Discrete)	63,893.43079588	SNS (20,000)	**63,893.43079587**	5 × 2000	63,893.43079640

Table 18 Set 2: comparison of results of test 2: constrained engineering problems; mean values

S. no	Problem	Mean	Algorithm (FEs)	TP-AB (no tuning)	PS * IT	TP-AB (with tuning)
1	Cantilever Beam	**1.3399576**	SNS (12,000)	1.3399584	10 × 5000	1.3399620
2	I-shaped beam	**0.0130743**	SNS (3600)	0.0130748	5 × 360	0.0130744
3	Three-bar truss	**263.895843**	AOS (100,000)	268.948347	5 × 10,000	267.685277
4	Tubular column	26.48636249	SNS (1250)	**26.48636147**	10 × 750	**26.48636147**
5	Speed reducer	**2994.465397**	CGO (100,000)	3006.525760	5 × 1000	2999.796617
6	Piston lever	24.3189743	SNS (5000)	**8.412702018**	5 × 500	**8.416034965**
7	Corrugated bulkhead	6.842979802	SNS (3125)	**6.842958020**	10 × 1000	6.843030016
8	Pressure vessel	**6068.05**	WOA (6300)	6366.020442	10 × 1500	6089.814189
9	Tension/compression spring (continuous)	**0.012684717**	SNS (9000)	0.013153041	5 × 900	0.012850106
10	Welded beam	1.724880	SNS (9000)	1.724938	10 × 1500	**1.724868**
11	Gear train	**1.6800E−10**	WSA (50,000)	1.03753e − 9	10 × 1250	4.57194e−10
12	Reinforced concrete beam	**359.3222001**	SNS (1000)	360.3392669	5 × 100	360.1206000
13	Car side impact	**22.85858**	CS (20,000)	22.981410	5 × 2000	23.008457

1.1111e + 02, **4.7155e−02**, 0, − 4.5361e−09, − 7.6923e −01, −2.2581e + 00, 0]. That is, the sixth constraint is violated here also.

Dhadwal et al. (2014) in their paper analysed this problem for 5,000 FEs using their "Advanced Particle Swarm Assisted Genetic Algorithm" (PSGA) and reported a cost of 64,578.374336 with all constraints satisfied. When TP-AB (no tuning) is run for the same number of FEs after considering a penalty value of 10^9, the cost obtained is 64,335.258579463246860541403293610 satisfying all the constraints, which is better than the result of PSGA. The variables and constraint values obtained are:

Variables, X = [3.0, 60.0, 3.1, 55.0, 2.6, 51.0, 2.2250836295992595381676437682472, 44.500279819685950144503294723108, 1.7498965164556807838636132146348, 34.993921685890605033364408882335].

Constraints, G = [−1.4115e−01, − 3.8306e + 02, − 6.9151e + 02, − 1.2034e+03, − 1.1111e+02, − 1.0134e−05, − 2.2908e−03, − 6.2594e−04, − 3.8462e−01, −2.2581e+00, 0].

However, when the mean values are considered, TP-AB reports a slightly higher value of 67,490.771940 as against 66,667.003313 of PSGA.

In another simulation, when the number of function evaluations is increased and results are obtained for 6 problems (Table 19) from Set 2, the results improve significantly. For these six problems considered; except the "Mean" value in the Three-Truss" problem, and "Min" values for the "Tension/ Compression Spring (continuous)" and "Car Side Impact" problems, TP-AB reports the best results in all other cases where "Mean" values are involved. That is, the best "Mean" values are reported by TP-AB (with and without tuning) in "Speed Reducer", TP-AB (no tuning) in "Pressure Vessel", TP-AB (with tuning) in "Tension/ Compression Spring (continuous)", TP-AB (no tuning) in "RCC Beam" and TP-AB (no tuning) in "Car Side Impact" problems.

Two more popular algorithms, the AOA and TLBO are tried for five problems for the same number of PS and FEs the results of which are presented in Table 20. Since the AOA evaluates the cost value only once per iteration, the number of iterations is doubled of AOA when compared with TLBO and TP-AB.

The computing environment, problem codes and the number of trials are kept the same for this analysis and the individual execution times are recorded.

TLBO and TP-AB report better values in four cases each whereas, the relative performance of AOA is moderate for these five problems. The execution time taken by TP-AB is less than TLBO but, higher than AOA.

5.3 Two more constrained real-world engineering problems

The results for the Set 3 problems are presented in Table 21 and the summary in Table 22. For the spring problem, Deb and Goyal's GeneAS algorithm reported a cost of 2.665 with constraints satisfied. The Firefly algorithm (Gandomi et al., 2011) was run for 50,000 FEs and reports a minimum cost of 2.658575665. However, FP-AB (with and without tuning) reports a minimum cost of 2.65855917 which is better than the Firefly algorithm in less number of FEs. The stepped cantilever beam is the same as the one described in the 14th problem of Set 2 except that all the variables are continuous. MATLAB help centre reports a minimum cost value of 63,408.9 (mean value: 70,409.2). The problem was solved by using GA with a population of 150 and 200 iterations (30,000 FEs). When TP-AB is used with a population size of 5 and run for 3000 iterations, the minimum volume obtained by FP-AB (no tuning) is 63,110.60215584. All 11 constraints are satisfied and report values less than zero.

Table 19 Results a few problems in set 2 after changing the FEs

Problem	TP-AB (no tuning)	
	Min	Mean
Three-bar truss	**263.89584338157447973571834154427**	263.8958607887005314296402502805
Speed reducer	**2994.4244657567364811256993561983**	**2994.4244657567364811256993561983**
Pressure Vessel	6059.7143350495989579940214753151	**6060.9036662117077867151238024235**
Tension/compression spring (continuous)	0.012667284079845436695066496213258	0.012710043715328070076941990862451
Reinforced concrete beam	**359.2080**	**359.30940000000009604264050722122**
Car side impact	**22.842986888450038662767838104628**	**22.844076629706414394149760482833**

Problem	TP-AB (with tuning)		Modified, PSxIt
	Min	Mean	
Three-bar truss	263.89584340941104301236919127405	**263.89584564682564860049751587212**	20 × 2500
Speed reducer	**2994.4244657567364811256993561983**	**2994.4244657567364811256993561983**	20 × 2500
Pressure Vessel	**6059.7143350484357142704539000988**	6069.8807705489571162615902721882	20 × 2500
Tension/compression spring (continuous)	**0.012665392496729131061039552719194**	**0.012674978504014782973441377578183**	20 × 1000
Reinforced concrete beam	**359.2080**	359.30940001451619991712504997849	20 × 1000
Car side impact	22.843040315222499003766642999835	22.844284016931617031787027372047	20 × 2500

Table 20 Best Results obtained by AOA, TLBO and Time Taken for Execution in seconds

S. no	Problem	AOA		TLBO		TP-AB (with tuning)	
		Best cost	Time, s	Best cost	Time, s	Best cost	Time, s
1	Welded beam	1.867918	5.511001	**1.724852**	6.254272	**1.724852**	5.902146
2	Spring	0.012846536	1.823609	**0.012665241**	2.165562	0.012665312	2.055374
3	Pressure Vessel	6312.785457	4.527306	**6059.714335**	5.408230	**6059.714335**	4.998653
4	3-Bar Truss	263.904095	17.399008	**263.895843**	20.919388	**263.895843**	20.381898
5	Speed Reducer	3062.035749	3.328771	2994.424588	3.556876	**2994.424466**	3.501252
	Total		**32.589695**		38.304328		36.839323

Table 21 Results of Set 3: two more constrained engineering problems

No	Problem	Min	Mean
	With tuning		
1	Tension/compression spring (discrete)	2.6585591659696001798351971956436	2.7486134556551009566760512825567
X = [9.0, 0.283, 1.2230410099638073795347281702561]			
G = [− 1.0088e + 03, − 8.9456e + 00, − 8.3000e − 02, − 1.4940e + 00, − 1.3217e + 00, − 5.4643e + 00, − 8.9456e + 00, − 4.4409e − 16]			
2	Cantilever stepped beam (continuous)	63,110.845732644156669266521930695	63,881.342620670984615571796894073
X = [3.0610794383168822285767873836448, 61.220771294622053915190917905 42, 2.8097138670718835307127392297843, 56.194100520533645237719611031935, 2.5236014430557296428503377683228, 50.471575720812950294202892109752, 2.2045843886269258504739809723105, 44.090890085904263173688377719373, 1.7504570444708369425512728412286, 34.988142408557607154762081336 23]			
	No tuning		
1	Tension/compression spring (discrete)	2.6585591659695992916567774955183	2.7719856354084919658475882897619
X = [9.0 0.283, 1.2230410099638071574901232452248]			
G = [1.0088e + 03, − 8.9456e + 00, − 8.3000e − 02, − 1.4940e + 00, − 1.3217e + 00, − 5.4643e + 00, − 8.9456e + 00, 0]			
2	Cantilever stepped beam (continuous)	63,110.602155842083448078483343124	64,259.179646525059069972485303879
X = [3.0518369934581484059776812500786, 61.036199273662191444600466638803, 2.8184006454484151760198074043728, 56.367483814648203122033010004088, 2.5250247088588966271061053703306, 50.500196772016529678239749046043, 2.2047225187226273668272824579617, 44.094166940239183816174772800878, 1.7499110125437822294003353817 97, 34.994745459319346991833299398422]			

No	Max	SD	PS*It
	With tuning		
1	3.2033490448868686684136264375411	0.14685732655674482716179340968665	5 × 600
2	71,033.937113035950460471212863922	1503.8341685265982050623279064894	5 × 3000

Table 21 (continued)

No	Max	SD	PS*It
	No tuning		
1	3.4872744652369740414599164068932	0.19033949210856959677329314217786	5 × 600
2	76,917.394928211477235890924930573	2688.4603952829784248024225234985	5 × 3000

Table 22 Set 3: Results of two more constrained engineering problems: TP-AB with and without tuning parameter [30 trials]

		No tuning parameter			With tuning parameter		
S. no	Problem	TP-AB (best)	TP-AB (mean)	PS * IT	TP-AB (best)	TP-AB (mean)	PS * IT
1.	Tension/compression spring (Discrete)	**2.65855917**	2.77198564	5 × 600	**2.65855917**	**2.74861346**	5 × 600
2	Cantilever stepped beam (Continuous)	**63,110.60215584**	64,259.17964653	5 × 3000	63,110.84573264	**63,881.34262067**	5 × 3000

Table 23 100 digit challenge: basic test functions (CEC2019)

S. no.	Function	Optimal value	No. of variables	Search range
1	Storn's Chebyshev polynomial fitting problem	1	9	[− 8192, 8192]
2	Inverse Hilbert matrix problem	1	16	[− 16,384, 16384]
3	Lennard–Jones minimum energy cluster	1	18	[− 4, 4]
4	Rastrigin's function	1	10	[− 100, 100]
5	Griewangk's function	1	10	[− 100, 100]
6	Weierstrass function	1	10	[− 100, 100]
7	Modified Schwefel's function	1	10	[− 100, 100]
8	Expanded Schaffer's F6 function	1	10	[− 100, 100]
9	Happy cat function	1	10	[− 100, 100]
10	Ackley function	1	10	[− 100, 100]

5.4 Results of CEC2019 100 digit challenge

Ten problems of CEC2019 (Table 23) popularly known as the "100-digit challenge" are analysed (Table 24). The details including the evaluation criteria and definition of problems could be found in the published Technical Report (Price et al., 2018).

Mohammed and Rashid (2020) compared the results of a few algorithms including hybrid ones for this dataset that are reproduced in Table 25 along with TP-AB, AOA and SCA. The results of WOA-BAT (Wolf Optimization Algorithm hybridized with BAT Algorithm) are extracted from Mohammed et al. (2019). These results are compared with that of TP-AB since the number of function evaluations (FEs: 15,000, Trials: 30) is the same in both cases. If the important metric of "Mean" values is considered, TP-AB (with tuning) is accountable for the best results in half of the cases followed by WOA-BAT in two problems. In the case of "Standard Deviation" also, TP-AB (with tuning) reports better results in four problems and GWO and SCA come next with two cases each.

6 Conclusion and future work

This paper proposes one novel and simple two-phase trigonometric algorithm TP-AB for solving global optimization problems. It involves two simple updating strategies which could be combined with a suitable tuning parameter also that gives slightly different result sets. TP-AB is a two-phase algorithm in which the (mu + lambda) strategy is applied to update the populations in the first phase and, "Greedy Selection" in the second phase. Applying the "Greedy Selection" or, (mu + lambda) selection in both the phases could also be tried on

Table 24 Results of 100 digit challenge (CEC2019) [PS: 5; It: 1500]

No	No tuning				With tuning			
	Min	Mean	Max	SD	Min	Mean	Max	SD
1	3.7668e+04	4.1878e+04	5.0307e+04	2.8515e+03	3.8238e+04	**4.1383e+04**	4.6389e+04	**2.4818e+03**
2	18.3429	**18.3429**	18.3429	1.5764e−09	18.3429	**18.3429**	18.3429	**9.6581e−13**
3	13.7024	**13.7024**	13.7024	7.7751e−08	13.7024	**13.7024**	13.7024	**6.3818e−12**
4	3.9849	70.8321	200.0922	53.7544	7.8558	**47.1778**	91.2161	**23.1974**
5	2.0000	2.2545	3.2163	0.3112	2.0145	**2.2073**	2.8141	**0.1860**
6	6.6677	**11.2361**	13.4043	**1.4075**	5.5434	9.7577	11.9932	1.5494
7	26.1149	321.3333	767.7056	162.8349	1.0187	**191.3025**	497.1132	**114.2477**
8	2.6507	5.0386	6.1303	**0.5004**	2.4999	**4.2173**	5.3706	0.6264
9	3.3925	**3.5747**	4.3213	**0.2198**	3.3646	3.6798	4.2941	0.2348
10	1.0000	**19.9501**	21.6322	**4.6633**	1.0000	20.0268	21.4254	4.5222

Table 25 Comparison of the results of the 100 digit challenge (CEC2019)

F	WOA		WOA-GWO		GWO		WOA-BAT		SCA		AOA	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
1	2.10E+10	3.57E+10	**4.76E+04**	**5.19E+03**	2.13E+08	3.07E+08	7.60E+07	4.16E+08	7.06E+09	9.87E+09	2.435E+09	6.110E+09
2	1.84E+01	1.61E−02	1.83E+01	4.72E−04	1.83E+01	**3.04E−04**	**1.75E+01**	1.21E−01	1.85E+01	1.57E−01	2.045E+01	3.348E−01
3	1.37E+01	**7.23E−15**	1.37E+01	1.83E−05	1.37E+01	1.92E+00	**1.27E+01**	9.53E−04	1.37E+01	1.33E−04	1.370E+01	8.831E−04
4	3.48E+02	**1.72E+02**	**2.54E+02**	5.39E+02	3.01E+02	6.87E+02	2.12E+03	1.01E+03	1.70E+03	9.37E+02	1.051E+04	3.827E+03
5	3.03E+00	4.86E−01	**2.43E+00**	2.62E−01	**2.43E+00**	2.52E−01	2.44E+00	6.67E−01	3.22E+00	**1.56E−01**	4.652E+00	8.626E−01
6	1.03E+01	1.39E+00	1.14E+01	1.64E+00	1.19E+01	**7.31E−01**	1.11E+01	1.55E+00	1.15E+01	9.67E−01	**9.880E+00**	1.106E+00
7	6.14E+02	2.98E+02	5.88E+02	3.49E+02	5.35E+02	2.92E+02	6.06E+02	3.90E+02	7.76E+02	1.71E+02	**2.256E+02**	**1.281E+02**
8	6.03E+00	5.66E−01	5.59E+00	1.02E+00	**5.40E+00**	9.94E−01	5.72E+00	7.18E−01	6.12E+00	**4.27E−01**	5.503E+00	6.216E−01
9	5.93E+00	**6.85E−01**	**5.67E+00**	8.81E−01	1.47E+01	5.00E+01	2.28E+01	4.92E+01	1.42E+02	1.28E+02	8.352E+02	4.169E+02
10	2.13E+01	1.35E−01	2.16E+01	9.22E−02	2.15E+01	**6.85E−02**	2.12E+01	2.26E−01	**1.01E+01**	1.24E+00	1.662E+01	2.163E+00

different test suites that need to be investigated. Initially, the performance is analysed using 10 unconstrained benchmark functions with dimensions 30, 50 and 100 followed by 13 more unimodal and multimodal test instances (dimensions vary from 30 to 1000). The efficacy is further analysed using 10 constrained special functions and 16 constrained engineering problems that represent real-world benchmarks. The number of decision variables varies from 2 to 11 and constraints up to 11 for the constrained engineering problems. Finally, the tough "100-digit challenge" data set of CEC2019 is solved and compared with a few other results available in the literature. Total number of problems tested and analysed are 59 in number. The penalty method is used for constraint handling.

The results are compared with several popular and recent better-performing algorithms including DE, ITLBO, TLBO, AOA, SCA, SNS, CGO, AOS, FA, MCEO, CS, WCA, WSA and WOA. The results demonstrate the better performance of TP-AB in particular TP-AB without any tuning parameter for all sorts of problems like constrained, unconstrained and, real-world engineering problems with continuous, discrete and both continuous and discrete variables. This proves the potential of the new algorithm which is a fair competitor to other population-based algorithms. Since the new algorithm has the capability of solving real-world engineering problems, future work includes analysis of more problems in other domains of operations research like scheduling, facility location and multi-objective optimization problems.

Acknowledgements The authors are grateful to the Editor and the anonymous reviewers for their constructive comments and suggestions to improve the quality and presentation of this paper.

Data availability The codes used in this paper are not available online and can be provided to potential researchers on request.

Declarations

Conflict of interest A. Baskar, M. Anthony Xavior and P. Jeyapandiarajan declares that they have no conflict of interests. Andre Batako is one of the ISPEM Conference Co-Chairs. Anna Burduk is the ISPEM Conference Chair.

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