Asghari, A

Wonders of Multiplication Table

Wonders of Multiplication Table

Amir H. Asghari

This paper originated as a talk for school teachers, aimed at demonstrating that the multiplication table is more than a simple memory aid. Instead, it is a powerful tool for exploring mathematical patterns and plays an essential role in both the discovery and teaching of mathematics. The initial goal was to illustrate how the table could act as a bridge between arithmetic, algebra, and proof. However, as the author delved into the table, its intricate structure began to unfold, with each idea seamlessly leading to the next. This evolution transformed the table's utility far beyond its initial educational purpose. This paper chronicles that journey—from its beginnings to its culmination in Faulhaber's formula and Bernoulli numbers. Rather than immediately diving into symbolic mathematics, the paper lets the multiplication table gradually reveal its potential as a gateway to advanced mathematical concepts.

A Memory Aid

Fig 1. Kashi's table

With the advent of place-value notation, the multiplication table emerged. Historically, the multiplication table has appeared in various forms, typically extending to 9 (as seen in a treatise on arithmetic by the Persian mathematician Kashi [1]; Figure 1), and in modern contexts, it extends to 10 or 12. These versions vary in format; for example, some include or omit rows and columns for multiples of 1, while others add rows and columns for multiples of zero.

Despite its many formats, the multiplication table is primarily viewed as a memory aid and serves instrumental understanding rather than relational understanding (as per Skemp's distinction between "knowing how" and "knowing why" [2]). This perception has led to scrutiny, but the practical utility of the multiplication table has consistently supported its place in education. For instance, in England, it was reinstated in 2022 through the multiplication tables check (MTC), a statutory assessment requiring Year 4 pupils to fluently recall tables up to 12 x 12 [3].

This article aims to support relational understanding by bringing to the forefront one of the most overlooked structural aspects of the table, demonstrating how this respect for structure can foster discovery and promote deeper mathematical insight.

Additive Structure of the Multiplication Table

Multiplication is often introduced as repeated addition, making one key purpose of the multiplication table the consolidation of frequently used sums for future reference. However, it is essential to recognize that the *multiplication table is fundamentally a condensed addition table*, which may conceal certain additive structures. The following examples offer a glimpse into these structures, with the hope that readers will be inspired to explore further. For brevity, we may use tables of varying sizes throughout.

Sum of Natural Numbers

The sum of the first natural numbers—a concept famously associated with the young Gauss—can be discovered through the multiplication table. Rather than presenting the formula outright, learners can use the table to hypothesize and then prove it, creating a more authentic and engaging learning experience. Figure 2 represents some initial sums represented on the table.

Fig 2. $\sum n$

1	2	**3**	4	5	6	7	8	9	10
2	4	**6**	8	**10**	12	14	16	18	20
3	6	9	12	**15**	18	**21**	24	27	30
4	8	12	16	20	24	**28**	32	**36**	40
5	10	15	20	25	30	35	40	**45**	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Although familiar with the multiplication table, I was surprised when I first recognized this pattern. The structure of the sum reveals itself within the table, making it possible to predict the position of key cells and thus deduce a formula for the sum of the first natural numbers. Here, finding the formula no longer requires a “moment of genius”; the structure is already embedded in the table, so discovering the sum is as simple as reading from it.

Fig 3. $2\sum n$

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Interestingly, doubling each highlighted cell’s number would make the result even clearer and more surprising. See Figure 3. Each green cell in the multiplication table clearly represents the product of two consecutive numbers. Moving from the $n-th$ green cell to the next involves adding $2(n+1)$. Consequently, the value in the $n-th$ green cell,$\mathrm{n}(n+1)$, is twice the sum of the first n natural numbers. This beautifully illustrates the idea, often expressed in mathematics, that "seeing the same thing in different ways" can reveal deeper truths.

$$2(1+2+3+\cdots n)=\ \mathrm{n}(n+1),$$

Thus,

$$1+2+3+\cdots n=\frac{n(n+1)}{2}.$$

At this point, you might wish to recount the story of how the young Gauss derived this sum formula. However, despite its beauty, this approach cannot be generalized to sums of higher powers. Thus, while it exemplifies a moment of mathematical genius, it also highlights the limitations of such a “genius method” when we try to extend it beyond sums of natural numbers.

Sum of Squares

Ideally, we could apply Gauss’s approach directly to find the sum of the first squares, but this method doesn’t work here.

$$1^2+2^2+3^2+4^2+\cdots+n^2$$

$$n^2+(n-1)^2+(n-2)^2+\cdots+1$$

We must turn to an alternative approach. We begin with an initial identity, then apply it successively to each subsequent value.

$$(n+1)^3-n^3=3n^2+3\mathrm{n}+1$$

By combining these expressions and applying the formula we previously derived for the sum of natural numbers, we arrive at the following result after some algebraic simplification.

$$1^2+2^2+3^2+4^2+\cdots+n^2=\frac{2n^3+3n^2+n}{6}.$$

Polya [4, p.64] describes this approach as “efficient, clear, and short,” noting that, “the problem appeared difficult—we cannot reasonably expect a much clearer or shorter solution. There is, as far as I can see, just one valid objection: the solution appears out of the blue, pops up from nowhere. It is like a rabbit pulled out of a hat.”

The solution indeed seems to arise suddenly. Furthermore, it doesn’t reveal any inherent multiplicative structure in the result.

Now, let's explore the sum of squares using the multiplication table. When working with the sum of the first n natural numbers, it helped to work with a multiple of the sum rather than the sum directly. This was more an observation than a deliberate strategy, yet it proved effective. Following Polya's advice, we might call it a "trick" and try applying it here. He suggests, "Do you wish to know what is behind a trick? Try to apply the trick yourself and then you may find out" [4, p.64]. For the sum of squares, we multiply each value by 3, just as we previously multiplied by 2 for the sum of natural numbers. See Figure 4.

Fig 4. $3\sum n^2$

1	2	**3**	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	**15**	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	**42**	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	**90**	100

In the multiplication table, the yellow-highlighted cells appear in rows corresponding to the sum of natural numbers and columns numbered by odd numbers. This structure reflects the following relationship:

$$3(1^2+2^2+3^2+\cdots n^2)=(1+2+3+\cdots+n)(2n+1).$$

Alternatively, we can express the sum of squares as:

$$1^2+2^2+3^2+\cdots n^2=\frac{1}{3}.\frac{n(n+1)}{2}.(2n+1).$$

Moving from one yellow cell to the next visually illustrates the inductive step, making the table a valuable introduction to proof by induction. It is worth emphasizing that the multiplicative structure of the result is readily apparent. Even more promising, the table continues to offer insights when applied to the sum of cubes.

Sum of Cubes

The sum of cubes readily appears in the table as the square of the sum of natural numbers. However, let us continue with our plan and multiply each cell by 4. Among all possible factorizations, we aim to choose ones with discernible factors. The first factor in each cell in Figure 5 consists of even numbers greater than or equal to 4, so we can focus on the second factor in each cell. To discern its multiplicative structure, we again use the table (Figure 6). It turns out that the coloured cells are in the rows numbered with natural numbers, and the columns numbered with the sum of natural numbers:

Fig 5. $4\sum n^3$

4 × 1				
	6 × 6			
		8 × 18		
			10 × 40	
				12 × 75

Fig 6. $n\sum n$

1	2	3	4	5	6	7	8	9	10
2	4	**6**	8	10	12	14	16	18	20
3	6	9	12	15	**18**	21	24	27	30
4	8	12	16	20	24	28	32	36	**40**
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

$$4(1^3+2^3+3^3+\cdots+n^3)=2(\mathrm{n}+1).n(1+2+3+\cdots+n)$$

$$1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Once again, the cells can be used to demonstrate the inductive step, while the multiplicative structure of the result,

$$2(\mathrm{n}+1).n(1+2+3+\cdots+n),$$

readily lends itself to the inductive step.

Sum of Higher Powers

Like any tool, the multiplication table has its limitations. As we move to higher powers, it becomes increasingly cumbersome to use. Nevertheless, it offers valuable insights before reaching its limits. Take, for example, the sum of the fourth powers of natural numbers. Figure 7 illustrates five times the initial sums shown in the table.

Fig 7. $5\sum n^4$

	17	35	59	74
5	85			
14		490		
30			1770	
55				4070

The row numbers correspond to the sum of squares, while the column numbers represent six times the sum of natural numbers minus one. For simplicity, let's denote the sum of natural numbers by Δ_n (since they are known as triangular numbers for good reason)) and the sum of squares by $\square_n$:

$$5\sum_{i=1}^{n} i^4 = \square_n \times (6\Delta_n - 1)$$

Even if we initially overlook the regular appearance of triangular numbers in the resulting formula, the elegant simplicity of the sum of the fourth powers encourages us to revisit previous sums.

$$\sum_{i=1}^{n} i = \Delta_n, \qquad \sum_{i=1}^{n} i^3 = \Delta_n^2$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}\mathrm{n}(n+1)(2n+1) = \square_n, \qquad \sum_{i=1}^{n} i^4 = \frac{1}{5}\square_n(6\Delta_n - 1)$$

This observation hints at what Edwards [5, 6] coins as *Faulhaber polynomials*.

$$\sum n^r (r\ odd) = \text{a polynomial in } n(n+1)$$

$$\sum n^r (r\ even) = (2n+1) \times \text{a polynomial in } n(n+1)$$

Expressing even powers with $\square_n$ as factor, instead of $(2n+1)$, has the advantage of requiring fewer coefficients when identifying the corresponding polynomial (which itself is in terms of Δ_n). For example, let us find $\sum_{i=1}^{n} i^6$.

We know the answer will be a polynomial of degree 7: $\square_n$ has degree 3, Δ_n has degree 2, hence a second-degree polynomial in terms of Δ_n has degree 4 (if written in terms of n):

$$\sum_{i=1}^{n} i^6 = \square_n \left(a_0 + a_1\Delta_n + a_2{\Delta_n}^2\right)$$

For $\mathrm{n} = 1,\ 2,\ 3$ we have:

$$a_0 + a_1 + a_2 = 1$$

$$a_0 + 3a_1 + 3^2 a_2 = 13$$

$$a_0 + 6a_1 + 6^2 a_2 = \frac{397}{7}$$

We could multiply all equations by 7, as suggested by our analysis of the multiplication table, to avoid dealing with fractions immediately. Using Cramer's Rule, we see the impact of multiplying each equation by 7.

$$a_0 = \frac{7^2}{7^3} \times \frac{\begin{vmatrix} 7\times 1 & 1 & 1 \\ 7\times 13 & 3 & 3^2 \\ 397 & 6 & 6^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3^2 \\ 1 & 6 & 6^2 \end{vmatrix}} = \frac{1}{7} \times 1$$

And:

$$a_1 = -\tfrac{1}{7} \times 6, \quad a_2 = \tfrac{1}{7} \times 12.$$

So:

$$\sum_{i=1}^{n} i^6 = \frac{1}{7} \Box_n \left(12\Delta_n^2 - 6\Delta_n + 1\right)$$

For the sum of odd powers, factor of ${\Delta_n}^2$ appears in each result starting from the third power. So, for example, to determine $\sum_{i=1}^{n} i^5$ we need only to solve a simple system of two equations with two unknowns to find the coefficients of a first-degree polynomial in terms of Δ_n .The final result is as follows:

$$\sum_{i=1}^{n} i^5 = \frac{1}{3} \Delta_n^2 (4\Delta_n - 1)$$

Naturally, we aim to find a general formula rather than solving each case individually. This general formula, known as *Faulhaber's formula*, emerges unexpectedly from our initial exploration of the multiplication table.

Even to Odd and Vice Versa

It is well-known that we can derive the formula for even powers from add and vice versa. Edwards [6] building on his interpretation of Faulhaber [7], pursued an approach from even to odd. Conversely, Knuth [8] advocated an approach from odd to even. Thus, we might choose to find a formula for either the odd powers or the even powers. However, attempting both individually proves more effective when moving toward a unified formula encompassing both. Let us proceed with the even powers first.

Even Powers

$$\sum_{i=1}^{n} i^2 = \Box_n$$

$$\sum_{i=1}^{n} i^4 = \frac{1}{5} \Box_n (6\Delta_n - 1)$$

$$\sum_{i=1}^{n} i^6 = \frac{1}{7} \Box_n \left(12\Delta_n^2 - 6\Delta_n + 1\right)$$

We can encapsulate these equations in matrix form:

$$\begin{bmatrix} \sum_{i=1}^{n} i^2 \\ \sum_{i=1}^{n} i^4 \\ \sum_{i=1}^{n} i^6 \end{bmatrix} = \square_n \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & \frac{6}{5} & 0 \\ \frac{1}{7} & -\frac{6}{7} & \frac{12}{7} \end{bmatrix} \begin{bmatrix} 1 \\ \Delta_n \\ \Delta_n^2 \end{bmatrix}$$

For simplicity, let's call the matrix of coefficients of polynomials in Δ_n, P_{2k} . The 3×3 matrix above is P_6. The critical insight lies not within the matrix P_{2k} itself, but in its inverse, ${P_{2k}}^{-1}$.

$$P_6^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2^2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 5 & 7 \end{bmatrix}$$

$$\mathrm{P}_{2\mathrm{k}}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . \\ 0 & \frac{1}{2} & 0 & 0 & 0 & . \\ 0 & 0 & \frac{1}{2^2} & 0 & 0 & . \\ 0 & 0 & 0 & \frac{1}{2^3} & 0 & . \\ 0 & 0 & 0 & 0 & \frac{1}{2^4} & . \\ . & . & . & . & . & . \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & . \\ 1 & 5 & 0 & 0 & 0 & . \\ 0 & 5 & 7 & 0 & 0 & . \\ 0 & 1 & 14 & 9 & 0 & . \\ 0 & 0 & 7 & 30 & 11 & . \\ . & . & . & . & . & . \end{bmatrix}$$

The left diagonal matrix has $\frac{1}{2^{n-1}}$ in row $\mathrm{n} \geq 1$. Thus, the key insight lies in the matrix on the right. Several interesting patterns emerge within this matrix. For example:

- The odd numbers appear along the main diagonal.
- The sums of squares form the diagonal directly below the main diagonal.

These smaller patterns are helpful in determining specific matrix elements, as they reduce the number of entries that need to be calculated. Remarkably, a larger, encompassing pattern emerges among these micro-patterns.

The entries on the main diagonal are generated by the following formal power series (from the coefficient of x onwards):

$$\frac{1+x}{(1-x)^2} = 1 + 3\mathrm{x} + 5x^2 + 7x^3 + 9x^5 + \cdots$$

The next diagonal is generated by the following power series:

$$(1 + 3x + 5x^2 + 7x^3 + 9x^5 + \cdots)(x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots)$$

$$\frac{1+x}{(1-x)^2} \cdot \frac{x}{(1-x)^2}$$

The next diagonal follows the series:

$$(1 + 3x + 5x^2 + 7x^3 + 9x^5 + \cdots)(x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots)^2$$

If we add one row and one column to the matrix to include the constant 1 from the power series $\frac{1+x}{(1-x)^2}$, the matrix appears as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 3 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 5 & 0 & 0 & 0 & . \\ 0 & 0 & 5 & 7 & 0 & 0 & . \\ 0 & 0 & 1 & 14 & 9 & 0 & . \\ 0 & 0 & 0 & 7 & 30 & 11 & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

This is the *Riordan array* $\left(\frac{1+x}{(1-x)^2}, \frac{x}{(1-x)^2}\right)$, organized differently. In the *Riordan array* $\left(\frac{1+x}{(1-x)^2}, \frac{x}{(1-x)^2}\right)$ [10; from here on, sequences in The On-Line Encyclopaedia of Integer Sequences will be cited only by their entry number; The current one is: A111125], entries are filled column by column instead of diagonal by diagonal:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . \\ 3 & 1 & 0 & 0 & 0 & 0 & . \\ 5 & 5 & 1 & 0 & 0 & 0 & . \\ 7 & 14 & 7 & 1 & 0 & 0 & . \\ 9 & 30 & 27 & 9 & 1 & 0 & . \\ 11 & 55 & 77 & 44 & 11 & 1 & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

The rearranged Riordan array corresponds to Edwards's reading [6] of Tits' [10] expansion of $x^r(x+1)^{r+1} - x^{r+1}(x-1)^r$ in matrix form:

$$\begin{bmatrix} \sum_{i=1}^{n} i^2 \\ \sum_{i=1}^{n} i^4 \\ \sum_{i=1}^{n} i^6 \\ . \end{bmatrix} = \frac{1}{2}(2n+1). \begin{bmatrix} 3 & 0 & 0 & 0 & . \\ 1 & 5 & 0 & 0 & . \\ 0 & 5 & 7 & 0 & . \\ 0 & 1 & 14 & 9 & . \\ . & . & . & . & . \end{bmatrix}^{-1} \begin{bmatrix} u \\ u^2 \\ u^3 \\ u^4 \\ . \end{bmatrix}$$

Edwards uses u to denote $2\sum n$ (i.e., $2\Delta_n$). So, we can transform the matrix form presented by Edwards into the format introduced in this paper:

$$\begin{bmatrix} \sum_{i=1}^{n} i^2 \\ \sum_{i=1}^{n} i^4 \\ \sum_{i=1}^{n} i^6 \\ . \end{bmatrix} = \square_n \left(\frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2^3} & 0 \\ . & . & . & . & . \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & . \\ 1 & 5 & 0 & 0 & . \\ 0 & 5 & 7 & 0 & . \\ 0 & 1 & 14 & 9 & . \\ . & . & . & . & . \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ \Delta_n \\ \Delta^2{}_n \\ \Delta^3{}_n \\ . \end{bmatrix}$$

Edwards denotes the matrix of interest as $G = \{g_{ij}\}$, and by reading the coefficients from the expansion $x^r(x+1)^{r+1} - x^{r+1}(x-1)^r$, gives the following formula for each entity of the matrix:

$$g_{ij} = \binom{i+1}{2(i-j)+1} + \binom{i}{2(i-j)+1}$$

But he overlooks the fact that:

$$\binom{i+1}{2(i-j)+1}+\binom{i}{2(i-j)+1}=\frac{2j+1}{2(i-j)+1}\binom{i}{2(i-j)}$$

Thus, the right side of the equation, expressed in terms of k for the row number and s for the column number, yields the formula of the entries of the *Riordan array* $\left(\frac{1+x}{(1-x)^2},\frac{x}{(1-x)^2}\right)$:

$$\mathrm{T}(k,s)=\frac{2k+1}{2s+1}\binom{k+s}{2s}$$

Odd Powers

Using a similar approach, we can derive a matrix identity for the odd powers:

$$\begin{bmatrix}\sum_{i=1}^{n} i^3\\ \sum_{i=1}^{n} i^5\\ \sum_{i=1}^{n} i^7\\ \vdots\end{bmatrix}=\Delta^2{}_n\left(\frac{1}{2}\begin{bmatrix}1&0&0&0&.\\0&\frac{1}{2}&0&0&.\\0&0&\frac{1}{2^2}&0&.\\0&0&0&\frac{1}{2^3}&.\\.&.&.&.&.\end{bmatrix}\begin{bmatrix}2&0&0&0&.\\1&3&0&0&.\\0&4&4&0&.\\0&1&10&5&.\\.&.&.&.&.\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\ \Delta_n\\ \Delta^2{}_n\\ \Delta^3{}_n\\ .\end{bmatrix}$$

This identity is equivalent to the one provided by Edwards, who denotes the matrix of interest as $\mathrm{F}=\{f_{ij}\}$:

$$\begin{bmatrix}\sum_{i=1}^{n} i^3\\ \sum_{i=1}^{n} i^5\\ \sum_{i=1}^{n} i^7\\ \vdots\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2&0&0&0&.\\1&3&0&0&.\\0&4&4&0&.\\0&1&10&5&.\\.&.&.&.&.\end{bmatrix}^{-1}\begin{bmatrix}u\\ u^2\\ u^3\\ u^4\\ .\end{bmatrix}$$

By considering the expansion $[x(x+1)]^r-[x(x-1)]^r$, Edwards [7] gives the following formula for each element of the matrix:

$$f_{ij}=\binom{i+1}{2(i-j)+1}$$

Surprisingly, these are also the coefficients of the following power series:

$$(1+2x+3x^2+4x^3+\cdots)^n$$

This series represents the expansion of $\left(\frac{1}{(1-x)^2}\right)^n$.

Now, we are only three steps away from finding a general formula for the sum of powers. The first step is to find the inverses of G and F.

Inverses

Both matrices, G and F , share the same structure: they are lower triangular matrices with 1's and 0's below the diagonal in identical positions.

Among the various ways to approach the inverse of these matrices, the most direct method proves to be the most insightful.

To handle both matrices simultaneously, let's consider a matrix A of the following form:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & a_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 & 0 & 0 \\ 0 & 1 & a_{43} & a_{44} & 0 & 0 & 0 \\ 0 & 0 & a_{53} & a_{54} & a_{55} & 0 & 0 \\ 0 & 0 & 1 & a_{64} & a_{65} & a_{66} & 0 \\ 0 & 0 & 0 & a_{74} & a_{75} & a_{76} & a_{77} \end{bmatrix}$$

Let B denote the inverse of A:

$$B = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & 0 & 0 & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & 0 & 0 \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} & 0 \\ b_{71} & b_{72} & b_{73} & b_{74} & b_{75} & b_{76} & b_{77} \end{bmatrix}$$

Since B is also lower triangular, solving BA = I for B provides a recursive equation for each row.

Row 1:

$$a_{11}b_{11} = 1$$

Row 2:

$$a_{11}b_{21} + b_{22} = 0$$

$$a_{22}b_{22} = 1$$

Row 3:

$$a_{11}b_{31} + b_{32} = 0$$

$$a_{22}b_{32} + a_{32}b_{33} = 0$$

$$a_{33}b_{33} = 1$$

And so on.

Each system can be solved by back substitution, beginning with $a_{ii}b_{ii} = 1$. However, let's focus on the first unknown in each system: b_{11}, b_{21}, b_{31}, b_{41} , ..., where a surprising pattern emerges more readily. To avoid the cumbersome array of indices and symbols involved in back substitution, let us use Cramer's rule for each system of equations, focusing on the first unknown.

Observing that the determinants of each row's coefficient matrix are products of odd numbers, we focus on the numerators in Cramer's Rule to derive the sequence: $c_1, c_2, c_3, \ldots$.

$$c_1 = 1$$

$$c_2 = \begin{vmatrix} 0 & 1 \\ 1 & a_{22} \end{vmatrix} = -1$$

$$c_3 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & a_{22} & a_{32} \\ 1 & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ a_{22} & a_{32} \end{vmatrix} = -a_{32}.c_2$$

$$c_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{32} & 1 \\ 0 & 0 & a_{33} & a_{43} \\ 1 & 0 & 0 & a_{44} \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ a_{22} & a_{32} & 1 \\ 0 & a_{33} & a_{43} \end{vmatrix} = -(a_{33}.c_2 + a_{43}.c_3)$$

Let us calculate c_5:

$$c_5 = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & a_{22} & a_{32} & 1 & 0 \\ 0 & 0 & a_{33} & a_{43} & a_{53} \\ 0 & 0 & 0 & a_{44} & a_{54} \\ 1 & 0 & 0 & 0 & a_{55} \end{vmatrix}$$

Expanding the determinant along the first column, we have:

$$c_5 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a_{22} & a_{32} & 1 & 0 \\ 0 & a_{33} & a_{43} & a_{53} \\ 0 & 0 & a_{44} & a_{54} \end{vmatrix}$$

Moving the last column to the first position (which requires three swaps in a 4 by 4 matrix, this introducing a negative sign), we get:

$$c_5 = -\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{32} & 1 \\ a_{53} & 0 & a_{33} & a_{43} \\ a_{54} & 0 & 0 & a_{44} \end{vmatrix}$$

Because determinants are bilinear, we can rewrite c_5 as follows:

$$c_5 = -\left(a_{53}\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{32} & 1 \\ 1 & 0 & a_{33} & a_{43} \\ 0 & 0 & 0 & a_{44} \end{vmatrix} + a_{54}\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{32} & 1 \\ 0 & 0 & a_{33} & a_{43} \\ 1 & 0 & 0 & a_{44} \end{vmatrix}\right)$$

Expanding the first determinant on the last row:

$$c_5 = -\left(a_{53}a_{44}\begin{vmatrix} 0 & 1 & 0 \\ 0 & a_{22} & a_{32} \\ 1 & 0 & a_{33} \end{vmatrix} + a_{54}\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{32} & 1 \\ 0 & 0 & a_{33} & a_{43} \\ 1 & 0 & 0 & a_{44} \end{vmatrix}\right)$$

Thus,

$$c_5 = -(a_{53}a_{44}c_3 + a_{54}c_4)$$

In a similar fashion, we can find c_6 and the other numbers in the sequence c_n.

$$c_6 = -(a_{55}a_{44}c_3 + a_{64}a_{55}c_4 + a_{64}c_5)$$

Before writing the general term for c_n, let us see how we can compute c's from matrix A itself. Suppose we want to find c_7. The coefficients that will appear in the final formula are underlined in the matrix:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & . \\ 1 & a_{22} & 0 & 0 & 0 & 0 & . \\ 0 & a_{32} & a_{33} & 0 & 0 & 0 & . \\ 0 & 1 & a_{43} & a_{44} & 0 & 0 & . \\ 0 & 0 & a_{53} & a_{54} & \underline{a_{55}} & 0 & . \\ 0 & 0 & 1 & a_{64} & a_{65} & \underline{a_{66}} & . \\ 0 & 0 & 0 & \underline{a_{74}} & \underline{a_{75}} & \underline{a_{76}} & a_{77} \end{bmatrix}$$

To find c_7, we multiply of the underlined numbers in the last row by their "corresponding" underlined numbers on the diagonal. Each product represents the coefficient of c indexed by the column number of the underlined number:

$$c_7 = -(a_{74}.a_{55}.a_{66}.c_4 + a_{75}.a_{66}.c_5 + a_{76}.c_6)$$

The $c_n's$ are recursively defined in terms of the previous c -values:

$$c_n = -\sum_{j=1}^{n-1}\left(\prod_{k=n-j+1}^{n-1} a_{kk}\right) a_{\overline{n(n-j)}}\, c_{n-j}, \text{ where } c_1 = 1 \text{ and } c_2 = -1$$

We also know that $a_{in} = 1$ for $i = 2n$, and $a_{in} = 0$ for $i > 2n$, which restricts the a -terms in the formula. This can be proved by induction, though it is a bit messy.

Let us now apply c_n when the a -terms comes from the matrix G.

$$c_1 = 1$$

$$c_2 = -1$$

$$c_3 = -a_{32}.c_2 = 5$$

$$c_4 = -(a_{33}.c_2 + a_{43}.c_3) = -63$$

$$c_5 = -(a_{53}a_{44}c_3 + a_{54}c_4) = 1575$$

Surprisingly the sequence of c_n has a direct formula [A004193], which is where *Bernoulli numbers* come into play.

$$c_n = \frac{2(2n+1)!\,B_{2n}}{n!\,2^n}$$

If we take a -terms from the matrix F, then c_n is as follows:

$$c_1 = 1$$

$$c_2 = -1$$

$$c_3 = -a_{32}.c_2 = 4$$

$$c_4 = -(a_{33}.c_2 + a_{43}.c_3) = -36$$

$$c_5 = -(a_{53}a_{44}c_3 + a_{54}c_4) = 600$$

Again, this gives us another appearance of Bernoulli numbers [A263445]:

$$c_n = (2n+1)(n+1)!\,B_{2n}$$

Let us now use the full potential of Cramer's rule and find the first column of the inverse of G, denoted by $\widehat{g_{n1}}$.

$$\widehat{g_{n1}} = (-1)^{n+1}\frac{2(2n+1)!\,B_{2n}}{n!\,2^n.3.5.\dots.(2n+1)} = 2B_{2n}$$

And the first column of the inverse of F:

$$\widehat{f_{n1}} = \frac{(2n+1)(n+1)!\,B_{2n}}{2.3.4.\dots.n.(n+1)} = (2n+1)B_{2n}$$

To understand the reason behind the surprising appearance of the Bernoulli numbers, it is easier to move to the next step, where we rewrite our findings in terms on n, rather than Δ_n and $\square_n$.

$\Delta_n, \square_n, n$

Recall that the matrices G and F appeared when we wrote the sum of even and odd powers based on Δ_n and $\square_n$, instead of directly based on n.

$$Even\ Powers = \square_n \left(\frac{1}{3}\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2^2} & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2^3} & 0\\ . & . & . & . & .\end{bmatrix}\begin{bmatrix}3 & 0 & 0 & 0 & 0\\ 1 & 5 & 0 & 0 & 0\\ 0 & 5 & 7 & 0 & 0\\ 0 & 1 & 14 & 9 & 0\\ . & . & . & . & .\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\ \Delta_n\\ {\Delta^2}_n\\ {\Delta^3}_n\\ .\end{bmatrix}$$

$$Odd\ Powers = {\Delta^2}_n \left(\frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2^2} & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2^3} & 0\\ . & . & . & . & .\end{bmatrix}\begin{bmatrix}2 & 0 & 0 & 0 & 0\\ 1 & 3 & 0 & 0 & 0\\ 0 & 4 & 4 & 0 & 0\\ 0 & 1 & 10 & 5 & 0\\ . & . & . & . & .\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\ \Delta_n\\ {\Delta^2}_n\\ {\Delta^3}_n\\ .\end{bmatrix}$$

Now we are about to write these identities based on n. This is a rather straightforward algebraic process with two surprising results.

What we need to do is to express $\square_n \begin{bmatrix}1\\ \Delta_n\\ {\Delta^2}_n\\ {\Delta^3}_n\\ .\end{bmatrix}$ and ${\Delta^2}_n \begin{bmatrix}1\\ \Delta_n\\ {\Delta^2}_n\\ {\Delta^3}_n\\ .\end{bmatrix}$ in terms of n. The following matrix form will handle both expressions at the same time.

$$\begin{bmatrix}\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & .\\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & .\\ 0 & 0 & \frac{1}{3.2} & 0 & 0 & 0 & .\\ 0 & 0 & 0 & \frac{1}{2^2} & 0 & 0 & .\\ 0 & 0 & 0 & 0 & \frac{1}{3.2^2} & 0 & .\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2^3} & .\\ . & . & . & . & . & . & \frac{1}{2^4}\end{bmatrix}\begin{bmatrix}2 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 3 & 2 & 0 & 0 & 0 & 0\\ 0 & 1 & 2 & 1 & 0 & 0 & 0\\ 0 & 1 & 4 & 5 & 2 & 0 & 0\\ 0 & 0 & 1 & 3 & 3 & 1 & 0\\ . & . & . & . & . & . & .\end{bmatrix}\begin{bmatrix}n\\ n^2\\ n^3\\ n^4\\ .\end{bmatrix}$$

This matrix product also forces us to bring the two identities that we had set aside so far:

$\sum 1 = n$ (this is given in the first row of the product)

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (this is given in the second row of the product)

The odd rows correspond to the *Pascal triangle*, simply because they represent the coefficients of n in ${\Delta^2}_n[1 \quad \Delta_n \quad {\Delta^2}_n \quad {\Delta^3}_n \quad .]^T$, which are determined by the coefficients of $n^k(n+1)^k$ that are the same as the coefficient of $(n+1)^k$.

The even rows correspond to the *Lucas triangle* [A029635], simply because they represent the coefficients of n in $\square_n[1 \quad \Delta_n \quad {\Delta^2}_n \quad {\Delta^3}_n \quad .]^T$, which are determined by the coefficients of $(2n+1)(n(n+1))^k$ that are the same as the coefficient of $(2n+1)(n+1)^k$.

Final Step

To consolidate our findings for both even and odd powers, we capture the following two matrix identities with a single unified identity:

$$\left(\frac{1}{3}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2^3} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 0 & 5 & 7 & 0 & 0 \\ 0 & 1 & 14 & 9 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}\right)^{-1} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 0 & 5 & 7 & 0 & 0 \\ 0 & 1 & 14 & 9 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}^{-1}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3.2 & 0 & 0 & 0 \\ 0 & 0 & 3.2^2 & 0 & 0 \\ 0 & 0 & 0 & 3.2^3 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\left(\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2^3} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 10 & 5 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 10 & 5 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}^{-1}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 & 0 \\ 0 & 0 & 2^3 & 0 & 0 \\ 0 & 0 & 0 & 2^4 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The following matrix identity consolidates all our findings. Notice that *starting from column 0*, the even columns encode the sums of the corresponding even powers (e.g., column 2 corresponds to the sum of squares), while the odd columns encode the sums of the corresponding odd powers.

$$\begin{bmatrix} \sum_{i=1}^n 1 \\ \sum_{i=1}^n i \\ \sum_{i=1}^n i^2 \\ \sum_{i=1}^n i^3 \\ \sum_{i=1}^n i^4 \\ \sum_{i=1}^n i^5 \\ \sum_{i=1}^n i^6 \\ \sum_{i=1}^n i^7 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{5} & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & \cdot \\ 0 & 0 & \frac{1}{21} & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & 0 & \cdot \\ 0 & 0 & 0 & \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 3.2 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 2.2 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.2^2 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.2^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \times$$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & \frac{1}{3.2} & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \frac{1}{2^2} & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \frac{1}{3.2^2} & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2^3} & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3.2^3} & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2^4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 4 & 5 & 2 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 5 & 9 & 7 & 2 & 0 & \cdot \\ 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ \cdot \end{bmatrix}$$

The two diagonal matrices simplify to half of the identity matrix. Therefore:

$$\begin{bmatrix} \sum 1 \\ \sum i \\ \sum i^2 \\ \sum i^3 \\ \sum i^4 \\ \sum i^5 \\ \sum i^6 \\ \sum i^7 \\ \cdot \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & -\frac{1}{15} & 0 & \frac{1}{5} & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & \cdot \\ 0 & 0 & \frac{1}{21} & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & 0 & \cdot \\ 0 & 0 & 0 & \frac{1}{6} & 0 & -\frac{1}{3} & 0 & \frac{1}{4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 4 & 5 & 2 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 5 & 9 & 7 & 2 & 0 & \cdot \\ 0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ \cdot \end{bmatrix}$$

If I had written this paper as a mathematician, perhaps I could have simply stated that if we write the inverse of the properly signed *second Pascal matrix* (as Edwards [6] coined the matrix below) as a product of two matrices—one holding the signs and the fractional part, and the other holding the natural part of the coefficient—we would obtain the two matrices above. Of course, the first matrix should be multiplied by $\frac{1}{2}$ .In fact, this is exactly what the approach in this paper has done by rethinking all the sums in terms of Δ_n and $\square_n$.

$$\begin{bmatrix} \sum^{n} n \\ \sum n^2 \\ \sum n^3 \\ \sum n^4 \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdot \\ -1 & 2 & 0 & 0 & 0 & \cdot \\ 1 & -3 & 3 & 0 & 0 & \cdot \\ -1 & 4 & -6 & 4 & 0 & \cdot \\ 1 & -5 & 10 & -10 & 5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}^{-1} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ \cdot \end{bmatrix}$$

Conclusion

This paper started with the modest objective of showing the importance of the multiplication table beyond just being a memory aid. However, one idea led to another, and eventually, it revealed a finer structure underlying Faulhaber's formula and Bernoulli numbers. Nonetheless, the excitement surrounding the latter should not overshadow the importance of the former.

The multiplication table is a valuable tool for experiencing real mathematics, as it is connected and beautiful. It can demystify some of the more idiosyncratic moments of genius and make mathematics more accessible and discoverable. Even in the simplest cases of summing powers up to cubes—commonly used in calculus when introducing definite integrals—the multiplication table suggests a more interconnected approach compared to the “rabbit-pulled-out-of-the-hat” method often used in calculus classes for centuries.

Finally, this paper also reflects the personal experience of the author in working on it. The author is a mathematics educator who has extensively studied how people, both in educational and historical contexts, think mathematically. This paper documents how the author himself thinks mathematically. Thus, the decision was made to write it as it unfolded, hoping that it will benefit both mathematics teachers and those researching mathematics education.

References

1. Aydin, Nuh, Ghada Bakbouk, and Lakhdar Hammoudi. *Al-Kāshī's Miftāḥ Al-Ḥisab*. Birkhauser, 2019.
2. Skemp, Richard R. "Relational understanding and instrumental understanding." *Mathematics teaching* 77.1 (1976): 20-26.
3. Multiplication Tables Check. https://www.gov.uk/government/collections/multiplication-tables-check.
4. Polya, George. *Mathematical discovery, 1962*. John Wiley & Sons, 1962.
5. Edwards, Anthony William F. "Sums of powers of integers: a little of the history." *The Mathematical Gazette* 66.435 (1982): 22-28.
6. Edwards, Anthony William F. "A quick route to sums of powers." *The American mathematical monthly* 93.6 (1986): 451-455.
7. Faulhaber, J. *Academia Algebrae,* 1961.
8. Knuth, Donald E. "Johann Faulhaber and sums of powers." *Mathematics of Computation* 61.203 (1993): 277-294.
9. OEIS Foundation Inc. Entry A111125 in The On-Line Encyclopaedia of Integer Sequences, https://oeis.org/A111125, 2024.
10. Tits, L. "Sur la sommation des puissances numériques." *Mathesis* 37 (1923): 353-355.

Dedication

In preparing this paper, I frequently consulted Edwards' work [5]. During one of my recent visits, I noticed something I had somehow missed before: his paper begins in the middle of the journal page where the preceding paper ends, marked by the names and affiliation of its author—none other than my late PhD supervisor, David Tall! David sadly passed away on 15 July 2024. I wish to dedicate this paper to his proud memory, as he has been the most influential figure in shaping my academic character.