Edwards BS, Ward MB. Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly*. 2004; 111(5): 411–424. Available from: doi:10.1080/00029890.2004.11920092.

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1. INTRODUCTION.

Let us start with a very elementary problem.
What fraction of the circle is shaded?

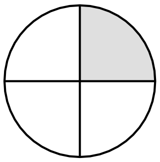

Figure 1: What fraction of the circle is shaded?

Now answer the following question.
In how many points does the line cross the circle?

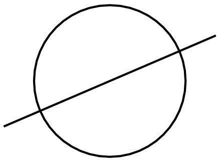

Figure 2: In how many points does the line cross the circle?

In moving from the first question to the second, as readers of *Monthly*, you can quickly switch from the everyday definition of a circle, also known as its *concept image*—"the mental attributes associated with [it]" [1, p. 152]—in which there is no distinction between a circle and a disk, to the mathematical definition of a circle, also known as its *concept definition*, which calls only for the set of points equidistant from a center. Thus, your answer to the first question is $\frac{1}{4}$, without any need to explicitly evoke the definition of a circle, or even to scrutinize your concept image; and your answer to the second question is 2, obtained by swiftly shifting to a concept image that is now in harmony with the formal concept definition of a circle.

However, it is well documented—for example, in the paper of interest by Edwards and Ward [2], among many others reviewed in [3]—that this shift between concept image and concept definition does not happen as swiftly for learners of mathematics as for mathematicians, for several reasons that we will discuss later.

In entering the fascinating world of concept image and concept definition, it is acceptable to think of the concept image as a mental image—but be aware that it is not necessarily an image per se; rather, it "includes all the mental pictures and associated properties and processes" [1, p. 152].

It is also acceptable to think of the concept definition as the mathematical definition of the concept expressed in words. Again, note that this definition could be a personal construction or reconstruction of a given definition, not necessarily the formal definition of the concept itself.

2. A HISTORY OF SURPRISES.

Edwards and Ward's paper was published in 2004, twenty-three years after the seminal work of Tall and Vinner [1] and thirteen years after Vinner's contribution [4], which together established the distinction between concept definition and concept image in the teaching and learning of mathematics, especially in advanced mathematical thinking [5] and in situations where reliance on formal definitions was essential.

Thus, when Edwards and Ward began their collaboration, Edwards—already a researcher in undergraduate mathematics education who had completed her PhD on students' understanding and use of definitions in undergraduate real analysis—was well accustomed to these ideas. However, Ward was not. As a mathematician invested in the teaching of mathematics, Ward challenged Edwards, betting that his students would have fewer—or at least different—difficulties with definitions in abstract algebra. The words there, such as group and coset, are "not so loaded," as opposed to terms like limit and continuity, which are burdened by connotations from their non-mathematical use and from their less-than-completely rigorous role in elementary calculus. Ward lost the bet—hence, the surprises in the title of the paper.

In what follows, I provide a review of Edwards and Ward's paper, then reflect on their work in light of what has been learned over the past twenty years, and conclude by extending their suggested implications for teaching.

3. SURPRISES AS LESSONS.

Ward's challenge was quite on point. Most previous mathematics education research had examined students' concept images in relation to definitions that they have already encountered. Ward therefore posed a different question: What if we begin with concepts whose definitions are entirely new to students, and whose terms carry little prior conceptual baggage? In such cases, he conjectured, students would have no alternative but to form and regulate their concept images through the given definitions. Alas, there were alternative possibilities.

Edwards and Ward interviewed eight students enrolled in Ward's Introduction to Group Theory course. In two individual task-based interviews, the students were given written definitions—either already introduced or about to be introduced in the course—which remained available throughout. The tasks required students to determine whether a given object satisfied a specified definition. For example, after reading the (new) definition of coset multiplication, students were asked to decide whether a quotient set forms a group under that operation. Edwards and Ward were particularly interested in whether and how students used the given definitions in addressing the tasks. They summarized their findings as three surprises, which I report here as three lessons, with some additional illustrative examples and commentary.

Lesson 1. Students need to learn that definitions in mathematics determine meaning and usage.

A reader might reasonably be surprised by the first lesson, asking whether it is not obvious that a definition determines meaning and usage. For learners of mathematics, however, this is not necessarily the case. Indeed, it is precisely this observation that gives rise to the notion of concept image: "During the mental processes of recalling and manipulating a concept, many associated processes are brought into play, consciously and unconsciously affecting the meaning and usage" [1, p. 152]. This insight motivated Edwards and Ward's choice of concept (coset), aiming to work with a definition carrying no "connotation on which to build a concept image independent of the definition" [2, p. 417]—or so they hoped.

In practice, however, the level of abstraction is often reduced in various ways to ease the cognitive demands of working with abstract concepts [6]. For instance, quotient groups are commonly represented using a division symbol; they could, in principle, be named in many different ways, but are instead called quotient or factor groups [7], choices that reflect how their relationship to less abstract ideas is perceived. This is precisely what occurred in Edwards and Ward's study. Lacking familiarity with the newly defined coset multiplication, participants reduced it to more familiar operations, interpreting it either as FOIL (First–Outer–Inner–Last) or as union, neither of which proved particularly helpful.

Lesson 2. Students need to learn that definitions in mathematics are not necessarily articulations of their concept image.

"You have to make the definitions from what something actually is," said one of the participants [2, p. 415]. But it is the definition that says what it actually is, not the other way around. However, bringing this point home is not easy. One classic case, also mentioned in the paper, is nought-dot-recurring. Many students accept its equality to 1 formally, yet this does not mean, for them, that it is actually equal to 1.

The distinction between what something *is* and what something *does* reflects a two-thousand-year overturn in mathematical thought. A case in point is the historical difficulty surrounding negative numbers. The resistance did not arise from computational inconvenience but from the question of what such numbers are: how could there be something less than zero? Acceptance came only when negative numbers were understood through their role: numbers that, when added to positive ones, produce zero.

It is conceptually challenging that, in mathematics, relations bring objects into existence, whereas elsewhere objects are typically assumed to exist first. Since mathematics taught up to calculus often remains shaped by the conception that definitions are expected to articulate pre-existing images, Lesson 2 has proved particularly challenging to enact. Still, the next lesson may offer a stepping stone.

Lesson 3. Students need to learn that there is a trade-off between concept image and concept definition.

"If necessary, [I] should have been able to extract the definition from the instances," remarked one of the participants [2, p. 416]. This reflects a familiar pedagogical strategy: motivating a definition by working from examples. In introducing equivalence relations, for instance, one might begin with a relation that partitions a set and one that does not, and then ask what makes one work while the other fails (see, e.g., [8]).

Properties we take for granted may lose their intuitive footing: transitivity requires three objects, symmetry two, and a relation such as $\{(a, a), (b, b), (c, c)\}$ on $\{a, b, c\}$—though an equivalence relation—sits uneasily with common experience [9]. Historically, equivalence was experienced by the property "if two objects are equivalent to a third, then they are equivalent to each other" [10], a formulation indistinguishable from transitivity in contextual examples such as parallel lines or congruence, but one that would reorganize how relations are classified [9]. None of this matters, of course, when one simply replaces a number by an equivalent one in congruence arithmetic.

In this sense, there is some truth in the calculus teacher's voice recalled by a participant in Edwards and Ward's study: "Now, the formal definition may mean a lot of jargon, but this is what it really means." [2, p. 421]

4. IMPLICATIONS AND CONCLUSION (AS EDWARDS AND WARD GAVE THEM).

Edwards and Ward continue with three implications, followed by a conclusion that is consistent with all three, but serves mainly to highlight the first: the special nature of mathematical definitions should be treated as a concept in its own right (Implication 1). They warn readers to be prepared to face students who have been conditioned by courses in which ignoring definitions is closer to the norm. As they note, this transition can be difficult and puzzling for many students (Implication 2). Thus, the third implication concerns preparing future teachers. If teachers are better equipped to address the role of definitions and precise language in mathematics [11], students need not encounter this only in a proof-based university course (Implication 3).

Across these implications, Edwards and Ward also propose a number of instructional activities, drawn from their own teaching practice. These include a Lakatos-style play [12] with constructing a definition for triangles on a sphere for which the Side–Angle–Side theorem holds [2], and a Socratic dialogue [13], aimed at settling on a definition of a quadrilateral, initiated simply by the question: What is a quadrilateral? [2, p. 419]. On some occasions, they also assigned readings from Vinner's work [4] so that the terminology of concept image and concept definition could be used explicitly in communication. For example, when an unjustified claim was made in a proof, one could ask whether it was based on concept image or on concept definition [2, p. 420].

5. REFLECTION.

Edwards and Ward's paper was grounded in an already-substantial literature on students' conceptions of mathematical definitions. Since then, research in this area has continued, resulting in a fairly robust body of knowledge synthesized by researchers worldwide (see, for example, [3]). By contrast, research on mathematicians' conceptions-in-action of definitions remains comparatively sparse ([14, 15, 16]). But we have a small but rich collection of retrospective accounts in which mathematicians open a window onto their own thinking ([17, 18]).

The above two bodies of work share a common thread. In mathematical practice, concept definition and concept image are not opposing forces, but different sides of the same coin. It is for this reason, I have softened the emphasis Edwards and Ward place on definitions when reporting their surprises. Rather than framing their observations as highlighting what mathematicians do with definitions that students do not, I have interpreted their observations as pointing to an interplay between concept image and concept definition—one that students need to learn.

For example, their third surprise is stated as follows: Many students do not use definitions the way mathematicians do, even in the apparent absence of any other course of action [2, p. 417]. But as they observed, there is always another course of action. The point is to learn how to control the connection between concept definition and concept image. Thus, the more productive approach is not whether a claim is based on concept image or concept definition, but when and why one should "listen" to the concept image, and when and why to the concept definition.

William Thurston [18, p. 5] points toward this interplay when he reflects on the need to "listen" to intuition and associations, while deliberately quieting words and formal logic that may inhibit the formation of connections. Jacques Hadamard [17, p. 77] offered a vivid illustration. Reflecting on his own imagery for the proof of the infinitude of primes, he insisted that such imagery is not meant to convey information about divisibility or prime numbers—since that would likely be inaccurate and misleading—but rather to guide how the elements of the argument should be brought together.

As Rupnow and Fukawa-Connelly [15] report, professional mathematicians often articulate a clear awareness of the importance of balancing concept image and concept definition in their own practice, and can describe personal ways of achieving this balance. However, when asked how such forms of understanding might be developed in students, their responses tended to remain vague, frequently amounting to little more than assigning further exercises or homework.

6. TEACHING REMARKS.

The list below—neither exhaustive nor mutually exclusive—suggests some ideas for a healthy interaction between concept image and concept definition. None of the items require doing anything additional; rather, they invite us to use familiar tools with an added purpose.

- **Non-example–looking examples** (e.g., a transitive relation with only two elements).
- **Example–looking non-examples** (e.g., the harmonic series).
- **Proof-generated definitions**, allowing students to experience defining objects in terms of what we want them to do rather than what they are. For instance, the historical interplay between concept image and concept definition in collective mathematical practice that led to the definition of uniform convergence [12].
- **Definitions within a broader organizational scheme**, where a locally workable definition may become limiting in a wider context, such as defining a trapezium so as to exclude or include parallelograms [19].

- **What definitions do not capture**, for example, the fact that the formal definition of limit offers a static formulation of a fundamentally dynamic idea.

Any of these situations can give rise to potential "talk" between elements of concept image and concept definition. Classroom settings are particularly well suited to evoke such talk, as different aspects of a concept can be brought into play simultaneously; for example, asking in how many points a line and a circle meet often surfaces competing interpretations. When anticipated and planned for, such moments can allow teachers to guide students in bridging their concept image and concept definition. Importantly, examples of this kind are not tied to proof alone and can equally be used in school mathematics.

We have come a long way since Edwards and Ward's study. The substantial body of research on students' concept images across a wide range of mathematical concepts suggests that we have not been successful in transmitting our own concept images to students. This echoes a remark made to me by Siavash Shahshahani—one of the most influential mathematics lecturers in Iran—in a personal interview: even lecturers with deep intuition often struggle to pass that intuition on directly, not least because intuition is difficult to put into words and requires considerable skill to communicate successfully. Even when such communication succeeds, the most we can hope for is that students come to appreciate our perspectives. As Terence Tao [20] observes, one's internal mathematical model is very much a function of one's mathematical upbringing; while one may appreciate other perspectives, one tends to revert to those with which one is most personally comfortable.

What we can reasonably hope for, and actively support, is that students form their own concept images, in harmony with the concept definition—both personal and public—while appreciating those aspects of our own concept image that can be articulated. Meanwhile, for researchers, it may now be time to focus more directly on understanding the interplay between concept image and concept definition.

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